Math 3330

**PRACTICE FINAL EXAM**

1. Consider the system of linear equations

\[
\begin{align*}
    x + 2y - w &= 2 \\
    2x + 3y - z + w &= 4 \\
    -y - z + 3w &= 0
\end{align*}
\]

Use Gaussian elimination to find all solutions (if any), keeping track of your elementary row operations. Indicate pivot and non-pivot variables. Express your answer in parametric form and give the translation and spanning vectors.

2. In the space \( \mathcal{F}(\mathbb{R}) \) of real-valued functions, consider the subset \( W = \{ f | f(2) = 0 \} \) (that is, the subset of functions such that \( f(2) = 0 \)). Show that \( W \) is a subspace of the space \( \mathcal{F}(\mathbb{R}) \).

3. Is the following set of vectors linearly independent:

\[
\begin{bmatrix}
1 \\
2 \\
-1 \\
1
\end{bmatrix},
\begin{bmatrix}
2 \\
-3 \\
-2 \\
5
\end{bmatrix},
\begin{bmatrix}
0 \\
1 \\
2 \\
3
\end{bmatrix}?
\]

4. Let \( A = \begin{bmatrix}
1 & 2 & 0 & -1 \\
2 & 3 & -1 & 1 \\
0 & -1 & -1 & 3
\end{bmatrix} \).

   (a) Find the basis of the row space of \( A \).
   (b) Find the basis of the column space of \( A \).
   (c) What is the rank of \( A \)?
   (d) What is the dimension of the null space of \( A \)?

5. Consider the function \( T : \mathbb{R}^2 \to \mathbb{R}^3 \) defined by \( T : \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y - 1 \\ x - y \\ 3y \end{bmatrix} \).

   Prove or disprove that \( T \) is a linear transformation.

6. Let \( T : \mathbb{R}^4 \to \mathbb{R}^3 \) and \( S : \mathbb{R}^3 \to \mathbb{R}^1 \) be matrix transformations given by matrices \( B = \begin{bmatrix} 1 & 2 & 0 & -1 \\
2 & -2 & 0 & 1 \\
0 & -1 & -1 & 3 \end{bmatrix} \) and \( A = \begin{bmatrix} 1 & 2 & 0 \\
0 & -1 & -1 \\
-1 & -1 & 3 \end{bmatrix} \), respectively. Find the matrix which represents the composition \( S \circ T \).

7. Consider the ordered basis \( B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\} \) of \( \mathbb{R}^3 \). Let \( X = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \).

   Find the coordinate vector of \( X \) in the basis \( B \).

8. Let \( A = \begin{bmatrix} 2 & 5 & -3 \\
2 & 1 & 1 \\
1 & -2 & 1 \end{bmatrix} \).

   (a) Compute the determinant of \( A \).
   (b) Determine if \( A \) is invertible and if so, compute the \((1, 2)\)-entry of its inverse.
9. Use Cramer’s rule to solve the system
\[
\begin{align*}
2x + 5y - 3z &= 1 \\
2x + y + z &= 0 \\
x - 2y + z &= 0
\end{align*}
\]

10. Let \( A = \begin{bmatrix} 3 & -2 & 0 \\ 2 & -2 & 0 \\ 0 & -1 & -1 \end{bmatrix} \).

   (a) Find all eigenvalues for the matrix \( A \) and a system of linearly independent eigenvectors of \( A \).

   (b) Determine if \( A \) is diagonalizable. Explain why or why not.

11. (a) Show that the vectors \( \mathbf{P}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \), \( \mathbf{P}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \), \( \mathbf{P}_3 = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} \) form an orthogonal basis of \( \mathbb{R}^3 \).

   (b) Find the coordinates for the vector \( \mathbf{X} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \) with respect to the basis from part (a).

12. Use the Gram-Schmidt orthogonalization procedure to find an orthogonal basis for the subspace spanned by the vectors \( \mathbf{A}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \) and \( \mathbf{A}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \).