Example 5.9 (From the book, p. 174)

The two-cell box beam section shown in the figure is symmetrical about the y-axis. Assume that the sheets are ineffective in bending.

Solution:

First, we need to calculate the centroidal y and z axes. First, \( \bar{y} \) is not needed, so it will not be calculated. On the other hand, because symmetry, \( \bar{z} = 20 \text{cm} \). Then, to calculate the inertia,

\[
I_y = \sum A_i z_i^2 = (1 \text{cm}^2)(20 \text{cm})^2 + (1 \text{cm}^2)(-20 \text{cm})^2 + (2 \text{cm}^2)(20 \text{cm})^2 + (2 \text{cm}^2)(-20 \text{cm})^2 +
\]

\[
+ (3 \text{cm}^2)(20 \text{cm})^2 + (3 \text{cm}^2)(-20 \text{cm})^2
\]

\[
I_y = 4,800 \text{cm}^4
\]

Then, "cut" the sheets between stringers 1 and 6 and between 5 and 6, and set up the contours \( s_1 \) and \( s_2 \) as shown in the figure.
Each contour must start from the cut (the free edge). The positive shear flow direction is assumed to be in the contour direction. In the flowing, \( q_{ij} \) is used to denote the shear flow between stringer \( i \) and \( j \).

**Cell 1:**

\[
q_{61} = 0
\]

\[
Q'_{y,12} = A_1 z_1 = (1 \text{cm}^2)(20 \text{cm}) = 20 \text{cm}^3
\]

\[
q'_{12} = q_{61} - \frac{V_z}{I_y} Q'_{y,12} = 0 - \frac{4,800N}{4,800cm^4} \frac{20cm^3}{20cm^3} = -20 \frac{N}{cm}
\]

\[
Q'_{y,23} = A_2 z_2 = (1 \text{cm}^2)(-20 \text{cm}) = -20 \text{cm}^3
\]

\[
q'_{23} = q_{12} - \frac{V_z}{I_y} Q'_{y,23} = -20 \frac{N}{cm} - \frac{4,800N}{4,800cm^4} (-20cm^3) = 0
\]
Cell 2:

\[ V_z = 4800 \text{ N} \]

\[ Q_{y,63} = A_6 z_6 = (2 \text{ cm}^2)(20 \text{ cm}) = 40 \text{ cm}^3 \]

\[ q_6 = q_{63} - \frac{V_z}{I_y} Q_{y,63} = 0 - \frac{4800 \text{ N}}{4800 \text{ cm}^4} \cdot \frac{40 \text{ cm}^3}{40 \text{ cm}^3} = -40 \frac{\text{ N}}{\text{ cm}} \]

\[ Q_{y,34} = A_4 z_3 = (2 \text{ cm}^2)(-20 \text{ cm}) = -40 \text{ cm}^3 \]

And because,

\[ q_{34} = q_{33} + q_{63} - \frac{V_z}{I_y} Q_{y,34} = 0 - 40 \frac{\text{ N}}{\text{ cm}} - \frac{4800 \text{ N}}{4800 \text{ cm}^4} \cdot \frac{40 \text{ cm}^3}{40 \text{ cm}^3} = 0 \]

\[ Q_{y,45} = A_4 z_4 = (3 \text{ cm}^2)(20 \text{ cm}) = 60 \text{ cm}^3 \]

\[ q_{45} = q_{34} - \frac{V_z}{I_y} Q_{y,45} = 0 - \frac{4800 \text{ N}}{4800 \text{ cm}^4} \cdot \frac{60 \text{ cm}^3}{60 \text{ cm}^3} = -60 \frac{\text{ N}}{\text{ cm}} \]
The shear flow are completed by adding the constant shear flows $q_1$ and $q_2$ (see the figure) in the individual cells, respectively. The equations needed for determining $q_1$ and $q_2$ are obtained from the moment equation and the compatibility equation.

\[ V_1 \cdot 0 = 2A_1 q_1 + 2A_2 q_2 + q_{13}' (40 cm)(40 cm) + q_{34}' (40 cm)(40 cm) + q_{45}' (40 cm)(80 cm) - q_{63}' (40 cm)(40 cm) \]

where $A_1 = A_2 = (40 cm)(40 cm) = 1600 cm^2$

Substituting the numerical values of $q_{ij}'$, we obtain

\[ 0 = 3200(q_1 + q_2) + 1600(q_{13}' + q_{34}' - q_{63}') + 3200(q_{45}') \]

\[ 0 = 3200(q_1 + q_2) + 1600(0 + 0 - (-40)) + 3200(60) \]

\[ 0 = 3200(q_1 + q_2) + 64000 + 192000 \]

\[ q_1 + q_2 = \frac{-64000 - 192000}{3200} \]

\[ q_1 + q_2 = -80 \frac{N}{cm} \]

Compatibility Equation: The compatibility condition requires that the twist angle of Cell 1 must be equal to that of Cell 2.

\[ \theta = \frac{1}{2GA} \int_0^t q \, ds \]

\[ \theta_{CELL1} = \theta_{CELL2} \]
\[
\frac{1}{2GA_1} \left[ q_{61} \frac{t_1}{t_1} + q_{12} \frac{t_1}{t_1} + q_{23} \frac{t_1}{t_1} - q_{63} \frac{t_1}{t_1} \right] = \frac{1}{2GA_2} \left[ q_{56} \frac{t_1}{t_1} + q_{63} \frac{t_1}{t_1} + q_{34} \frac{t_1}{t_2} + q_{45} \frac{t_1}{t_2} \right]
\]

And,

\[q_{61} = q_{61}^* + q_1 = q_1\]
\[q_{12} = q_{12}^* + q_1 = q_1 - 20\]
\[q_{23} = q_{23}^* + q_1 = q_1\]
\[q_{63} = q_{63}^* - q_1 + q_2 = -40 - q_1 + q_2\]
\[q_{34} = q_{34}^* + q_2 = q_2\]
\[q_{45} = q_{45}^* + q_2 = 60 + q_2\]
\[q_{56} = q_{56}^* + q_2 = q_2\]
Substituting back,

\[
\frac{40}{2GA_t t_1} [q_{61} + q_{12} + q_{23} - q_{63}] = \frac{40}{2GA_t} \left[ \frac{q_{56}}{t_2} + \frac{q_{63}}{t_1} + \frac{q_{34}}{t_2} + \frac{q_{45}}{t_2} \right]
\]

\[
[q_{61} + q_{12} + q_{23} - q_{63}] = t_1 \left[ \frac{q_{56}}{t_2} + \frac{q_{63}}{t_1} + \frac{q_{34}}{t_2} + \frac{q_{45}}{t_2} \right]
\]

\[
[q_{61} + q_{12} + q_{23} - q_{63}] = \frac{t_1}{t_2} \left[ \frac{q_{56}}{t_1} + \frac{t_2}{t_1} q_{63} + \frac{q_{34}}{t_2} + \frac{q_{45}}{t_2} \right]
\]

\[
\left[ (q_1) + (q_1 - 20) + (q_1) - (-40 - q_1 + q_2) \right] = \frac{1}{2} \left[ (q_2) + 2(-40 - q_1 + q_2) + (q_2) + (60 + q_2) \right]
\]

\[
2[4q_1 - q_2 + 20] = \left[ (q_2) + (-80 - 2q_1 + 2q_2) + (q_2) + (60 + q_2) \right]
\]

\[
[8q_1 - 2q_2 + 40] = [5q_2 - 20 - 2q_1]
\]

\[
q_1(8 + 2) + q_2(-2 - 5) = -60
\]

\[
10q_1 - 7q_2 = -60
\]

Solving these system of equations,

\[
\begin{aligned}
q_1 + q_2 &= -80 \\
10q_1 - 7q_2 &= -60
\end{aligned}
\]

We get that,

\[
q_1 + q_2 = -80 \quad \Rightarrow \quad q_1 = -80 - q_2
\]

\[
10q_1 - 7q_2 = -60 \quad \Rightarrow \quad 10(-80 - q_2) - 7q_2 = -60 \quad \Rightarrow \quad -800 - 17q_2 = -60 \quad \Rightarrow
\]

\[
q_2 = \frac{-740}{17} = -43.53 \frac{N}{cm}
\]

\[
q_1 = -80 - q_2 \quad \Rightarrow \quad q_1 = -80 - (-43.53) \quad \Rightarrow \quad q_1 = -36.47 \frac{N}{cm}
\]
Shear Center: To find the shear center, we assume that the applied force passes through the shear center as shown in the figure. The resultant torque of the shear flow and the torque produced by $V_z$ must be equal. Taking the moment about stringer 1, we have,

\[ V_z e_y = 2 \bar{A}_1 q_1 + 2 \bar{A}_2 q_2 + q_{23} (40 \text{ cm})(40 \text{ cm}) + q_{34} (40 \text{ cm})(40 \text{ cm}) + q_{45} (40 \text{ cm})(80 \text{ cm}) - q_{65} (40 \text{ cm})(40 \text{ cm}) \]

\[ V_z e_y = 3,200(q_1 + q_2) + 64,000 + 192,000 \]

\[ V_z e_y = 3,200(q_1 + q_2) + 256,000 \]

By definition of shear center, we require that,

\[ \theta_1 = 0 = \left[ q_{41} + q_{12} + q_{23} - q_{63} \right] \]

\[ \theta_2 = 0 = \left[ q_{56} + \frac{t_2}{t_1} q_{63} + q_{63} + q_{63} \right] \]

These three equations are sufficient to solve for $q_1$, $q_2$, and the shear center location $e_y$. The solutions are

From the first equation,

\[ 4,800 e_y = 3,200(q_1 + q_2) + 256,000 \]

\[ q_1 + q_2 = \frac{4,800 e_y - 256,000}{3,200} = 1.5 e_y - 80 \]

From the second equation,
\[
[q_{61} + q_{12} + q_{23} - q_{63}] = 0
\]

\[4q_1 - q_2 + 20 = 0\]

From the third equation,

\[
\left[ q_{56} + \frac{t_2}{t_1} q_{63} + q_{63} + q_{63} \right] = 0
\]

\[5q_2 - 20 - 2q_1 = 0\]

Solving for \(q_1\) and \(q_2\) we get that,

\[4q_1 - q_2 + 20 = 0 \quad \Rightarrow \quad q_2 = 4q_1 + 20\]

\[5q_2 - 20 - 2q_1 = 0 \quad \Rightarrow \quad 5(4q_1 + 20) - 20 - 2q_1 = 0 \quad \Rightarrow \quad 20q_1 + 100 - 2q_1 = 0 \quad \Rightarrow \]

\[18q_1 = -80 \quad \Rightarrow \quad q_1 = \frac{-80}{18} = -4.44 \text{ cm}\]

\[q_2 = 4q_1 + 20 \quad \Rightarrow \quad q_2 = 4(-4.44) + 20 \quad \Rightarrow \quad q_2 = 2.22 \text{ cm}\]

Substituting back,

\[q_1 + q_2 = 1.5e_y - 80\]

\[-4.44 + 2.22 = 1.5e_y - 80\]

\[1.5e_y = -2.22 + 80\]

\[e_y = \frac{77.78}{1.5}\]

\[e_y = 51.85 \text{ cm}\]