Problem 3-1

The figure below shows a thin-walled tubular section of a wingbox composed of three cells. The internal shear flow pattern will be determined in resisting the external torque of 100,000lb-in as shown.
A) Find the angle of twist for this problem.
B) Solve the problem again but this time imagine the whole section is composed of only one cell.
C) Do the problem again but instead of one cells use two cells.
D) Compare the results.

![Diagram of tubular section with three cells]

Solution:

Part A) three cells section:

First, we need to calculate the cell constants. Let’s start with cell areas.

\[ A_1 = 39.3in^2 \quad A_2 = 100in^2 \quad A_3 = 100in^2 \]

Now the line integrals, \[ a = \oint \frac{ds}{t}, \]

\[ a_{10} = \frac{\pi \cdot d}{2t} = \frac{\pi \cdot 10in}{2(0.025in)} = 628 \]
\[ a_{12} = \frac{10\text{in}}{0.05\text{in}} = 200 \]
\[ a_{20} = \frac{20\text{in}}{0.03\text{in}} = 667 \]
\[ a_{23} = \frac{10\text{in}}{0.03\text{in}} = 333 \]
\[ a_{30} = \frac{20\text{in}}{0.03\text{in}} + \frac{10\text{in}}{0.04\text{in}} = 917 \]

Equating the external torque to the internal resisting torque:

\[ T = 2q_1A_1 + 2q_2A_2 + 2q_3A_3 \]

Substituting,

\[ (78.6)q_1 + (200)q_2 + (200)q_3 - 100,000 = 0 \]

Equation 1

Writing the expression for the angular twist of each cell,

For Cell 1:

\[ 2G\theta = \frac{1}{A_1} \left[ q_1a_{10} + (q_1 - q_2)a_{12} \right] \]

Substituting,

\[ 2G\theta = \frac{1}{39.3} \left[ (628)q_1 + (200)q_1 - (200)q_2 \right] \]

Equation 2

For Cell 2:

\[ 2G\theta = \frac{1}{A_2} \left[ (q_2 - q_1)a_{12} + q_2a_{20} + (q_2 - q_3)a_{23} \right] \]

Substituting,

\[ 2G\theta = \frac{1}{100} \left[ (200)q_2 - (200)q_1 + (667)q_2 + 333q_2 - (333)q_3 \right] \]

Equation 3
For Cell 3:

\[ 2G\theta = \frac{1}{A_3} [(q_3 - q_2)a_{23} + q_3a_{33}] \]

Substituting,

\[ 2G\theta = \frac{1}{100} [(333)q_3 - (333)q_2 + (917)q_3] \]  

Equation 4

Solving equations (1) to (4), we obtain,

- \( q_1 = 143.4 \text{ lb/in} \)
- \( q_2 = 234.1 \text{ lb/in} \)
- \( q_3 = 208.8 \text{ lb/in} \)
- \( q_2 - q_1 = 90.7 \text{ lb/in} \)
- \( q_2 - q_3 = 25.3 \text{ lb/in} \)

The figure shows the resulting internal shear flow pattern. The angle of twist can be found by substituting values of shear flows in any of the equations (2) to (4). From equation 4, we get,

\[ 2G\theta = \frac{1}{100} [(333)q_3 - (333)q_2 + (917)q_3] \]

\[ 2G\theta = \frac{1}{100} [(333)(208.8) - (333)(234.1) + (917)(208.8)] \]

\[ \theta = \frac{915}{G} \]

Note: "G" is the Shear Modulus. Also, the units of the angular twist \( \theta \) are \( \text{rad/in} \).
Part B) one cell section:

The area of this cell is just,

\[ A_i = 239.3 \text{in}^2 \]

And there is only one constant,

\[ a_{10} = \frac{\pi \cdot 10\text{in}}{2(0.025\text{in})} + \frac{40\text{in}}{0.03\text{in}} + \frac{10\text{in}}{0.04\text{in}} = 2212 \]

Again, equating the external torque to the internal resisting torque:

\[ T = 2q_1A_i \]

Substituting,

\[ (478.6)q_1 - 100,000 = 0 \]

From here we get that,

\[ q_1 = \frac{100,000}{478.6} = 208.9 \text{lb/in} \]

Now, writing the expression for the angular twist for the only cell we have,

\[ 2G\theta = \frac{1}{A_i}[q_1a_{10}] \]
Substituting,

\[ 2G \theta = \frac{1}{239.3} \left[ (2212)(208.9) \right] \]

\[ \theta = \frac{966}{G} \]

**Part C) two cells section:**

Now we have only two cells and their areas are,

\[ A_1 = 139.3 \text{in}^2 \quad A_2 = 100 \text{in}^2 \]

Their line integrals \( a = \int \frac{ds}{t} \) are,

\[ a_{10} = \frac{\pi \cdot 10 \text{in}}{2(0.025 \text{in})} + \frac{20 \text{in}}{0.03 \text{in}} = 1295 \]

\[ a_{12} = \frac{10 \text{in}}{0.03 \text{in}} = 333 \]

\[ a_{20} = \frac{20 \text{in}}{0.03 \text{in}} + \frac{10 \text{in}}{0.04 \text{in}} = 917 \]

Just as before, equating the external torque to the internal resisting torque:

\[ T = 2q_1 A_1 + 2q_2 A_2 \]
Substituting,

\[(279)q_1 + (200)q_2 - 100,000 = 0\] \hspace{1cm} \text{Equation 5}

Once again, writing the expression for the angular twist of each cell,

**For Cell 1:**

\[2G\theta = \frac{1}{A_1} [q_1 a_{1o} + (q_1 - q_2) a_{12}]\]

Substituting,

\[2G\theta = \frac{1}{139.3} [(1295)q_1 + (333)q_1 - (333)q_2]\] \hspace{1cm} \text{Equation 6}

**For Cell 2:**

\[2G\theta = \frac{1}{A_2} [(q_2 - q_1) a_{12} + q_2 a_{20}]\]

Substituting,

\[2G\theta = \frac{1}{100} [(333)q_2 - (333)q_1 + (917)q_2]\] \hspace{1cm} \text{Equation 7}

Solving the equations from (5) to (7), we get that,

\[q_1 = 208.0 \text{ lb/in}\]
\[q_2 = 209.8 \text{ lb/in}\]
\[q_2 - q_1 = 1.8 \text{ lb/in}\]

And,

\[\theta = \frac{965}{G}\]
Now arranging the two cell in a different configuration,

Again, we have only two cells and their areas are,

\[ A_1 = 39.3\text{in}^2 \quad A_2 = 200\text{in}^2 \]

Their line integrals \( a = \oint a \cdot ds \) are,

\[ a_{10} = \frac{\pi \cdot 10\text{in}}{2(0.025\text{in})} = 628 \]

\[ a_{12} = \frac{10\text{in}}{0.03\text{in}} = 333 \]

\[ a_{20} = \frac{20\text{in}}{0.03\text{in}} + \frac{20\text{in}}{0.03\text{in}} + \frac{10\text{in}}{0.04\text{in}} = 1583 \]

Just as before, equating the external torque to the internal resisting torque:

\[ T = 2q_1A_1 + 2q_2A_2 \]

Substituting,

\[ (78.6)q_1 + (400)q_2 - 100,000 = 0 \quad \text{Equation 8} \]
Once again, writing the expression for the angular twist of each cell,

For Cell 1:

\[ 2G \theta = \frac{1}{A_i} \left[ q_i a_{i10} + (q_1 - q_2) a_{i12} \right] \]

Substituting,

\[ 2G \theta = \frac{1}{39.3} \left[ (628)q_1 + (333)q_1 - (333)q_2 \right] \]

Equation 9

For Cell 2:

\[ 2G \theta = \frac{1}{A_2} \left[ (q_2 - q_1) a_{12} + q_2 a_{20} \right] \]

Substituting,

\[ 2G \theta = \frac{1}{200} \left[ (333)q_2 - (333)q_1 + (1583)q_2 \right] \]

Equation 10

Solving the equations from (8) to (10), we get that,

\[ q_1 = 152.1 lb/in \]
\[ q_2 = 220.1 lb/in \]
\[ q_2 - q_1 = 68.0 lb/in \]

And,

\[ \theta = \frac{928}{G} \]
Part D) comparing the results:

<table>
<thead>
<tr>
<th></th>
<th>3-cells</th>
<th>2-cells Configuration #2</th>
<th>2-cells Configuration #1</th>
<th>1-cell</th>
</tr>
</thead>
<tbody>
<tr>
<td>Angular Twist $\theta$</td>
<td>915/G</td>
<td>928/G</td>
<td>965/G</td>
<td>966/G</td>
</tr>
<tr>
<td>Difference with respect to 3-cells</td>
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<td>1.42%</td>
<td>5.46%</td>
<td>5.57%</td>
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</tbody>
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