A Constrained Plate under a Thermal Gradient

Square Plate:

As we know,

$$\varepsilon^{TOTAL} = \varepsilon^{MECHANICAL} + \varepsilon^{THERMAL}$$

In this case, the total strain is equal to "zero" because of the constrain.

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} + \alpha \Delta T = 0$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \nu \frac{\sigma_x}{E} + \alpha \Delta T = 0$$

From these two equations, we get that,

$$\frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} + \alpha \Delta T = \frac{\sigma_y}{E} - \nu \frac{\sigma_x}{E} + \alpha \Delta T$$

$$\sigma_x - \nu \sigma_y = \sigma_y - \nu \sigma_x$$
\[(1 + \nu)\sigma_x = (1 + \nu)\sigma_y\]

\[
\sigma_x = \sigma_y
\]

So, both stresses will be the same! Now, substituting back,

\[
\frac{\sigma_x}{E} - \nu \frac{\sigma_x}{E} + \alpha \Delta T = 0
\]

\[
\frac{\sigma_x}{E} - \nu \frac{\sigma_x}{E} = -\alpha \Delta T
\]

\[
(1 - \nu)\sigma_x = -E\alpha \Delta T
\]

\[
\sigma_x = \frac{E\alpha \Delta T}{(1 - \nu)} = \sigma_y
\]

So the stresses are independent of the geometry. That is, \(\sigma_x\) is the same for every point in the square plate; the same happens for \(\sigma_y\).

**Rectangular Plate:**

![Rectangular Plate Diagram]

Once again, the total strain will be equal to zero. Therefore,
\[ \varepsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} + \alpha \Delta T = 0 \]

\[ \varepsilon_y = \frac{\sigma_y}{E} - \nu \frac{\sigma_x}{E} + \alpha \Delta T = 0 \]

Solving again these two equations, we get that,

\[
\begin{align*}
\sigma_x &= \sigma_y \\
\sigma_x &= \frac{E \alpha \Delta T}{(1 - \nu)} = \sigma_y
\end{align*}
\]

**How is this possible?**

Well, under these conditions in the rectangular plate, the horizontal (\(\sigma_x\)) and vertical (\(\sigma_y\)) stresses will be the same. However, these stresses are acting on different areas; therefore, the forces are different. Remember that,

\[ P_x = \sigma_x A \]

Therefore, for the Square Plate,

\[ P_x^{\text{SQUARE}} = \sigma_x a \Delta z \]

On the other hand, for the Rectangular Square,

\[ P_x^{\text{RECT}} = \sigma_x a \Delta z = \left( \frac{E \alpha \Delta T}{(1 - \nu)} \right) a \Delta z \]

\[ P_y^{\text{RECT}} = \sigma_y b \Delta z = \left( \frac{E \alpha \Delta T}{(1 - \nu)} \right) b \Delta z \]

Finally,

\[ P_x^{\text{RECT}} = \frac{a}{b} P_y^{\text{RECT}} \]

**Note:** Remember that both forces are in compression; that is why they are both negative.