#1. Consider the following steady, two dimensional velocity field:

\[ V = (u, v) = (0.5 + 1.2x)i + (-2.0 - 1.2y)j \]

Is there a stagnation point in this flow field? If so where is it?

Solve: At stagnation point

\[ u=0 \Rightarrow 0.5 + 1.2x = 0 \Rightarrow x = -0.5/1.2 = -0.417 \]
\[ v=0 \Rightarrow -2.0 - 1.2y = 0 \Rightarrow y = -1.67 \]

So, the stagnation point is \((-0.417, -1.67)\)
#2. A steady, incompressible, two dimensional velocity field is given by the
following components in the x-y plane:

\[ u = 1.1 + 2.8x + 0.65y \]
\[ v = 0.98 - 2.1x - 2.8y \]

Calculate the acceleration field and calculate the acceleration the fluid
experiences at point \((x, y) = (-2, 3)\).

**Solution:**

\[ \dot{a} = \frac{\partial \mathbf{V}}{\partial t} + (\nabla \cdot \mathbf{V}) \mathbf{V} = u \frac{\partial \mathbf{V}}{\partial x} + v \frac{\partial \mathbf{V}}{\partial y} \]

\[ u \frac{\partial V_x}{\partial x} = u \frac{\partial}{\partial x} \left[ (1.1 + 2.8x + 0.65y) \mathbf{i} + (0.98 - 2.1x - 2.8y) \mathbf{j} \right] \]
\[ = u (2.8 \mathbf{i} - 2.1 \mathbf{j}) \]

\[ v \frac{\partial V_y}{\partial y} = v \frac{\partial}{\partial y} \left[ (1.1 + 2.8x + 0.65y) \mathbf{i} + (0.98 - 2.1x - 2.8y) \mathbf{j} \right] \]
\[ = v (0.65 \mathbf{i} - 2.8 \mathbf{j}) \]

\[ \therefore \dot{a} = (1.1 + 2.8x + 0.65y) (2.8 \mathbf{i} - 2.1 \mathbf{j}) + (0.98 - 2.1x - 2.8y)(0.65 \mathbf{i} - 2.8 \mathbf{j}) \]
\[ = (3.717 + 6.475x) \mathbf{i} - (5.054 - 6.495y) \mathbf{j} \]

At point \((x, y) = (-2, 3)\)

\[ \dot{a} = [3.717 + (6.475)(-2)] \mathbf{i} - [5.054 - (6.495)(3)] \mathbf{j} \]
\[ = -9.233 \mathbf{i} + 14.37 \mathbf{j} \]
#3. The temperature $T$, in a long tunnel is known to vary approximately as $T = T_0 - \alpha e^{-\frac{\pi}{L} \sin \left(\frac{2\pi t}{t}\right)}$, where $T_0$, $\alpha$, $L$ and $t$ are constants, and $x$ is measured from the entrance. A fluid moves into the tunnel with a constant speed, $U$. Obtain an expression for the rate of change of temperature experienced by the fluid.

**Solution:**

**Given:** $T = T_0 - \alpha e^{-\frac{\pi}{L} \sin \left(\frac{2\pi t}{t}\right)}$

\[
\frac{dT}{Dt} = \frac{\partial T}{\partial t} + (V \cdot \nabla) T
\]

\[
\frac{\partial T}{\partial t} = 0 - \alpha e^{-\frac{\pi}{L} \frac{2\pi t}{t}} \cos \left(\frac{2\pi t}{t}\right) = -\frac{2\pi}{t} \alpha e^{-\frac{\pi}{L} \cos \left(\frac{2\pi t}{t}\right)}
\]

\[
(V \cdot \nabla) T = U \frac{\partial T}{\partial x} = U \frac{\partial}{\partial x} \left[ T_0 - \alpha e^{-\frac{\pi}{L} \sin \left(\frac{2\pi t}{t}\right)} \right]
\]

\[
= \frac{U \alpha}{L} e^{-\frac{\pi}{L} \sin \left(\frac{2\pi t}{t}\right)}
\]

So, \[
\frac{dT}{Dt} = -\frac{2\pi}{t} \alpha e^{-\frac{\pi}{L} \cos \left(\frac{2\pi t}{t}\right)} + \frac{U \alpha}{L} e^{-\frac{\pi}{L} \sin \left(\frac{2\pi t}{t}\right)}
\]

\[
= \alpha e^{-\frac{\pi}{L}} \left[ \frac{U}{L} \sin \left(\frac{2\pi t}{t}\right) - \frac{2\pi}{t} \cos \left(\frac{2\pi t}{t}\right) \right]
\]
#4. Consider steady flow of air through the diffuser of a wind tunnel as shown. Along the centerline of the diffuser, the air speed decreases from $u_{\text{entrance}}$ to $u_{\text{exit}}$ as sketched. Measurements reveal that the centerline airspeed decreases parabolically through the diffuser. Write an equation for the centerline speed $u(x)$ and the fluid acceleration $a(x)$ along the centerline based on the parameters given, $x = 0$ to $x = L$. For $L = 2.0 \text{ m}$, $u_{\text{entrance}} = 30 \text{ m/s}$ and $u_{\text{exit}} = 5.0 \text{ m/s}$, calculate the acceleration at $x = 0$ and $x = 1.0 \text{ m}$.

Solve: According to centerline airspeed decrease parabolically,

$$u(x) = u_{\text{entrance}} - ax^2$$

At $x = 0$, $u = u_{\text{entrance}} = 30 \text{ m/s}$

$x = L = 2.0 \text{ m}$, $u = u_{\text{exit}} = 5.0 \text{ m/s}$

$$5 = 30 - aL^2 = 30 - 4a \implies a = \frac{25}{4} = 6.25$$

So, $u(x) = 30 - 6.25x^2$

$$a = u \frac{\partial u}{\partial x} = (30 - 6.25x^2)(-12.5x)$$

At $x = 0$, $a = 0$

$x = 1.0 \text{ m}$, $a = (30 - 6.25)(-12.5) = -296.875 \text{ m/s}^2$
# 5. After discarding any constants of integration, determine the appropriate value of the unknown velocities \( w \) or \( v \) that satisfy the equation of three-dimensional incompressible continuity for

\[
(a) \quad u = x^2 y z \quad v = -y^2 x \\
(b) \quad u = x + 3z \quad v = -z^2 + y
\]

Solve: three-dimensional incompressible continuity:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]

a) \[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \Rightarrow 2xyz - 2xy + \frac{\partial w}{\partial z} = 0 \]

\[ \Rightarrow w = \int (2xyz - 2xy) \, dz = xy z^2 - 2xyz \]

b) \[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \Rightarrow 2x + 3z^2 + \frac{\partial v}{\partial y} - 3z^2 = 0 \]

\[ \Rightarrow v = \int -2x \, dy = -2xy \]
#6. The forced vortex shown below is composed of streamlines that are concentric circles. The velocity is given by \( v_\theta = \omega r \) and \( v_r = 0 \) where \( \omega \) is the constant angular velocity of the vortex. Is this flow irrotational? Determine the circulation around path \( ABCD \).

\[ \mathbf{T} = \oint_{ABCD} \mathbf{V} \cdot d\mathbf{s} = \int_{AB} V_\theta b \, d\theta + \int_{BC} V_r \, dy + \int_{CD} V_\theta a \, d\theta + \int_{DA} V_r \, dy \]

Since \( V_r = 0 \), \( V_\theta = \omega r \)

\[ \mathbf{T} = \int_{\theta_1}^{\theta_2} \omega b^2 d\theta + \int_{\theta_1}^{\theta_2} \omega a^2 d\theta \]

\[ = \omega b^2 (\theta_2 - \theta_1) + \omega a^2 (\theta_2 - \theta_1) \]

\[ = \omega \Delta \theta (b^2 - a^2) \]

\[ \therefore \text{This flow is not irrotational} \]
#7. A two-dimensional flow is defined by

\[ u = -\frac{K_y}{x^2 + y^2}, \quad v = \frac{K_x}{x^2 + y^2} \]

Where \( K = \text{constant} \). Is this flow incompressible and irrotational? If so, find its velocity potential and stream function. Plot a few potential and stream lines, and interpret the flow pattern.

**Solve:** For incompressible flow, \( \nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \)

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{-2kxy}{(x^2 + y^2)^2} - \frac{2kx^2 y}{(x^2 + y^2)^2} = 0
\]

\( \therefore \) this flow is incompressible.

For irrotational flow, \( \nabla \times \vec{V} = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \)

\[
\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \frac{2ky^2 - k(x^2 + y^2)}{(x^2 + y^2)^2} + \frac{2kx^2 - k(x^2 + y^2)}{(x^2 + y^2)^2} = 0
\]

\( \therefore \) this flow is irrotational.

\[ u = \frac{\partial \phi}{\partial x} = -\frac{ky}{x^2 + y^2} \Rightarrow \phi = \int -\frac{ky}{x^2 + y^2} \, dx = -k \tan^{-1}\left(\frac{x}{y}\right) = k \theta \]

Flow pattern: source.
#8. A two-dimensional incompressible flow field is defined by the velocity components

\[
\begin{align*}
  u &= 2V\left(\frac{x}{L} - \frac{y}{L}\right) \\
  v &= -2V\frac{y}{L}
\end{align*}
\]

where \( V \) and \( L \) are constants. If they exist, find the stream function and velocity potential.

Solve: \( \nabla \cdot \vec{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{2V}{L} - \frac{2V}{L} = 0 \)

\( \therefore \) satisfy continuity means \( \psi \) exist.

\( \nabla \times \vec{v} = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = -\frac{2V}{L} - 0 = -\frac{2V}{L} \neq 0 \)

\( \therefore \) not satisfy irrotational means \( \phi \) does not exist.

According to: \( \frac{\partial \psi}{\partial y} = u = 2V\left(\frac{x}{L} - \frac{y}{L}\right) \)

\( \Rightarrow \psi = \int 2V\left(\frac{x}{L} - \frac{y}{L}\right) dy = \frac{2Vxy}{L} - \frac{Vy^2}{L} + C(x) \)

According to: \( \frac{\partial \psi}{\partial x} = -v = 2V\frac{y}{L} \)

\( \Rightarrow \psi = \int 2V\frac{y}{L} dx = 2V\frac{xy}{L} + C(y) \)

So, \( \psi = \frac{2Vxy}{L} - \frac{vy^2}{L} + C \)
#9. A circular cylinder is fitted with two surface-mounted pressure sensors, to measure \( p_a \) at \( \theta = 180^\circ \) and \( p_b \) at \( \theta = 105^\circ \). The intention is to use the cylinder as a stream velocimeter. Using potential flow theory, derive a formula for estimating \( U_\infty \) in terms of \( p_a, p_b, \rho, \) and the cylinder radius \( a \).

Solve: Tangential velocity on the surface of circular cylinder in free stream velocity \( U_\infty \) is

\[
V_\theta = -2U_\infty \sin \theta
\]

Apply Bernoulli equation between free stream and point a

\( \theta = 180^\circ \)

\[
-p_\infty + \frac{1}{2} \rho U_\infty^2 = p_a + \frac{1}{2} \rho U_a^2 = p_a + \frac{1}{2} \rho (-2U_\infty \sin 180^\circ)^2
\]

\[
\Rightarrow p_\infty = p_a - \frac{1}{2} \rho U_a^2
\]

(1)

Apply Bernoulli equation between free stream and point b

\( \theta = 105^\circ \)

\[
-p_\infty + \frac{1}{2} \rho U_\infty^2 = p_b + \frac{1}{2} \rho U_b^2 = p_b + \frac{1}{2} \rho (-2U_\infty \sin 105^\circ)^2
\]

\[
\Rightarrow p_\infty = p_b + 2 \rho U_\infty \sin^2(105^\circ) - \frac{1}{2} \rho U_b^2
\]

(2)

According to equation (1) & (2).

\[
p_a - \frac{1}{2} \rho U_a^2 = p_b + 2 \rho U_\infty \sin^2(105^\circ) - \frac{1}{2} \rho U_b^2
\]

\[
\Rightarrow p_a - p_b = 2 \rho U_\infty \sin^2(105^\circ)
\]

\[
\Rightarrow U_\infty = \sqrt{\frac{p_a - p_b}{2 \rho \sin^2(105^\circ)}}
\]
#10. A Rankine half-body is formed as shown. For the stream velocity and body dimension shown, compute (a) the source strength \( m \) in \( \text{m}^2/\text{s} \), (b) the distance \( a \), (c) the distance \( h \), and (d) the velocity magnitude at point A.

![Diagram of a Rankine half-body](image)

Solve for Rankine half body

\[
\psi = U r \sin \theta + m \theta, \quad m = \frac{a}{2\pi}
\]

At stagnation point, \( r = a \), \( \theta = \pi \)

\[
\psi = ma
\]

\[
V_r = U = \frac{m}{a} \Rightarrow a = \frac{m}{U}
\]

So the equation of streamline passing through the stagnation point is:

\[
a\pi U = Ur \sin \theta + aU \theta \quad \Rightarrow \quad r = \frac{a(\pi - \theta)}{\sin \theta}
\]

b) At point \((0, 3)\), \( r = 3 \), \( \theta = \frac{\pi}{2} \)

\[
a = \frac{3}{\frac{\pi}{2}} = \frac{6}{\pi} \Rightarrow \quad a = 1.91 \text{ m}
\]

a) \( m = aU = \left(\frac{6}{\pi}\right)(7) = 13.37 \text{ m/s}\)

c) At point \((4, 0)\), \( \theta = \frac{4}{\cos \theta} \)

\[
\frac{4}{\cos \theta} = \frac{1.9(\pi - \theta)}{\sin \theta} \Rightarrow \theta = 47.8^\circ
\]

\[
h = 4 \tan \theta = 4 \tan (47.8^\circ) = 4.44 \text{ m}
\]

d) \( V^2 = V_r^2 + V_\theta^2 = U^2 (1 + 2 \frac{a}{r} \cos \theta + \frac{a^2}{r^2}) \)

\[
= 7^2 (1 + 2 \frac{(1.91)}{4/\cos(47.8^\circ)} \cos(47.8^\circ) + \frac{(1.91)^2}{4/\cos(47.8^\circ)^2}) = (8.67)^2
\]

\[
V = 8.67 \text{ m/s}
\]
#11. Sketch the streamlines, especially the body shape, due to equal line sources \( +m \) at \((-a,0)\) and \((+a,0)\) plus a uniform stream \( U_\infty = ma \).

\[ \psi = \psi_{\text{uniform}} + \psi_{\text{two source}} \]

\[ = U_\infty y \sin \theta + m(\theta_1 + \theta_2) \]

(superposition)

The corresponding streamlines for this flow field can be obtained by setting \( \psi = \text{constant} \).
#12. A wind with $U_\infty$ and $p_\infty$ flows past a canopy. The canopy is modeled as a half-cylinder of radius $a$ and length $L$ into the paper as shown. The internal pressure is $p_i$. Using potential flow theory, derive an expression for the upward force on the canopy due to the difference between $p_i$ and $p_s$.

\[ \begin{align*}
\text{Solve: On the surface, } \ U_\theta &= -2U_\infty \sin \theta \\
\text{By Bernoulli equation, } \\
p_\infty + \frac{1}{2} \rho U_\infty^2 &= p_s + \frac{1}{2} \rho U_\theta^2 = p_s + 2U_\infty^2 \sin^2 \theta \\
&\Rightarrow \ p_s = \frac{1}{2} \rho U_\infty^2 (1 - 4 \sin^2 \theta) + p_\infty \\
dF &= p dA = -(p_s - p_i) \ a d\theta \cdot \hat{L} \\
dL &= dF \sin \theta = -(p_s - p_i) \sin \theta \ a \ L \ d\theta \\
L &= \int_0^\pi -aL \left[ \frac{1}{2} \rho U_\infty^2 (1 - 4 \sin^2 \theta) + p_\infty - p_i \right] \sin \theta \ d\theta \\
&= aL (p_i - p_\infty - \frac{1}{2} \rho U_\infty^2) \int_0^\pi \sin \theta d\theta - 2 \rho U_\infty^2 L a \int_0^\pi \sin^3 \theta d\theta \\
&= 2aL (p_i - p_\infty - \frac{1}{2} \rho U_\infty^2) + 2 \rho U_\infty^2 L a \cdot \frac{4}{3} \\
&= 2aL (p_i - p_\infty) + \frac{5}{3} \rho U_\infty^2 L a
\end{align*} \]
6.63. *Fundamentals of Fluid Mechanics, Munson*

#13. A spinning cylinder is placed in a uniform irrotational and incompressible flow as shown below. For what angular velocity \( \Omega \) will the stagnation point be located at:

a) point A  

b) point B

Which case produces the greatest lifting force?

Solve: Adding a free vortex to velocity potential for the flow around a cylinder:

\[ \phi = U \gamma \left( 1 + \frac{a^2}{r^2} \right) \cos \theta + \frac{\Omega r}{2\pi} \theta \]

So, the tangential velocity \( V_\theta \) on the surface of cylinder becomes:

\[ V_\theta = -\frac{\partial \phi}{\partial r} \bigg|_{r=a} = -2U \sin \theta + \frac{\Omega r}{2\pi a} \]

For stagnation point, \( V_\theta = 0 \)

\[ T = 4\pi U a \sin \theta \]

At point A: \( \theta = 0 \)  \( \Rightarrow T = 0 \)

At point B, \( \theta = \frac{\pi}{2} \), \( \Rightarrow T = -4\pi U a \)

\[ \tau = \frac{V_\theta}{a} = \frac{T}{2\pi a^2} = -\frac{4\pi U a \Omega}{2\pi a^2} = -\frac{2U}{a} \]

At point B: \( F_y = -\rho \tau = 4\pi \rho a U^2 \), the greatest lifting force.