1. (10 points) A 1:30 scale model of a ship is to be tested in a towing tank. Determine the required
kinematic viscosity of the model fluid so that both the Reynolds number and the Froude number are
the same for model and prototype. The prototype fluid is to be seawater at 60°F.

2. (20 points) For a steady, two-dimensional incompressible flow in \( x-y \) plane, show that the \( z \)
component of vorticity \( \zeta \) and the stream function \( \psi \) are related by the equation:

3. (20 points) A layer of incompressible fluid with a uniform thickness, \( h \), between a large plate and an
inclined surface moves steadily down parallel to the inclined
surface, as shown in the figure. The plate moves with a
constant velocity while the inclined surface is fixed at an
angle \( \theta \). The plate has a surface area of \( A \) and a mass of \( M \).
Assume the fluid has a density \( \rho \) and a constant viscosity \( \mu \);
the flow is steady and laminar. The acceleration of gravity, \( g \),
is a constant. Starting with the full Navier-Stokes equations,
derive an expression for the velocity of the plate, \( U \), in terms of
\( \mu, \rho, \theta, A, M, g, \) and \( h \).

4. (30 points) A turbulent Newtonian flow of liquid water through a horizontal section of pipe is shown
in the figure below. The pipe has a circular cross-section
with constant radius, \( r_0 \). The time-averaged fully developed
velocity profile is approximated as \( u(r) = u_{\max} \left( 1 - \frac{r^4}{r_0^4} \right) \).
Assume a steady flow with \( u_{\max} = 100.0 \, \text{m/s}, L = 500.0 \, \text{m} \), and
\( r_0 = 0.25 \, \text{m} \).
a. Calculate the mass flow rate through the pipe.
b. Calculate the differential height reading, \( h \), on the
manometre.
c. Calculate the reaction force in the \( x \)-direction required to hold the pipe in place.
d. If the flow is adiabatic, calculate the temperature change between the inlet and the exit?
e. Calculate the Reynolds number. Assume the reference length is \( r_0 \) and the reference velocity is the
average cross-section velocity.
f. If the water is replaced with an inviscid fluid with the same volumetric flow rate, how would the
answers to parts b.-e. change?

5. (20 points) A tank with the nozzle shown below is open to the atmosphere. The nozzle has a
length of \( L \) and constant radius of \( R \). The liquid exits the nozzle as a free jet into the
atmosphere. The desired volume flow rate for the nozzle is \( Q \). Find an expression for the
required height of the liquid in the tank, \( h \), in terms of the given variables under the following
assumptions:
a. inviscid flow in the nozzle
b. laminar Newtonian viscous flow in the
nozzle
Assume the tank is large enough that the
liquid height does not change with time.
Also assume the flow in the nozzle is steady
and incompressible. Acceleration of gravity
is \( g \).
Choose x-axis along the incline.

As eqn. in x-dir:

\[ \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + fg \]

\[ \Rightarrow \quad \frac{\partial p}{\partial x} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} + \rho g \sin \theta \]

Since thickness is uniform, \( h = \text{const} \). Therefore \( \frac{\partial^2 p}{\partial x^2} = 0 \)

So we have:

\[ \mu \frac{\partial^2 u}{\partial y^2} = -\rho g \sin \theta \]

Integrating twice gives:

\[ u = -\frac{\rho g}{2\mu} \sin \theta y^2 + c_1 y + c_2 \quad \text{—— (I)} \]

Boundary Conditions are:

\[ y = 0 \quad u = 0 \]

\[ y = h \quad u = U \]

First B.C. gives \( c_2 = 0 \)

Second B.C. gives \( c_1 = \frac{U}{h} + \frac{\rho g h \sin \theta}{2\mu} \)

\( \Rightarrow \quad u(y) = \frac{U}{h} + \frac{\rho g \sin \theta}{2\mu} (h - y) \quad \text{—— (II)} \)

Since plate is moving with constant velocity (\( \Rightarrow \text{Eqvm.} \))

Sine component of its weight is balanced by the shear force \( VA \).
\[ M \sin \theta = 2A \]

\[ z = \mu \frac{\partial u}{\partial y} \bigg|_{y=h} = \frac{M \sin \theta}{A} \]

\[ \frac{\partial u}{\partial y} = \frac{U}{h} + \frac{pgh \sin \theta}{2\mu} - \frac{pgh \sin \theta}{\mu} \]

\[ \frac{\partial u}{\partial y} \bigg|_{y=h} = \frac{U}{h} - \frac{pgh \sin \theta}{2\mu} \]

\[ z = \mu \frac{\partial u}{\partial y} \bigg|_{y=h} = \mu \left( \frac{U}{h} - \frac{pgh \sin \theta}{2\mu} \right) \]

Substituting this in (iii)

\[ \mu \left( \frac{U}{h} - \frac{pgh \sin \theta}{2\mu} \right) = \frac{M \sin \theta}{A} \]

\[ \frac{\mu U}{h} = \frac{M \sin \theta}{A} + \frac{pgh \sin \theta}{2} \]

\[ U = \frac{h}{\mu} \left[ \frac{M \sin \theta}{A} + \frac{pgh \sin \theta}{2} \right] \]

So

\[ U = \frac{gh \sin \theta}{\mu} \left( \frac{M}{A} + \frac{ph}{2} \right) \]

Any
\( \rho = 999 \text{ kg/m}^3 \)
\( \gamma = 9.8 \text{ kN/m}^3 \)
\( \kappa = 1.12 \times 10^{-5} \text{ Ns/m}^2 \)
\( C_v = 4.187 \text{ kJ/kgK} \)

\( u(\xi) = u_{\text{max}} \left( 1 - \left( \frac{\xi}{\xi_0} \right)^6 \right) \)

\[
\dot{m} = \int \rho u_n \cdot dA = \int_0^{\xi_0} \rho u_{\text{max}} \left( 1 - \left( \frac{\xi}{\xi_0} \right)^6 \right) 2\pi \eta_0 \, d\xi
\]

\[
= \rho u_{\text{max}} \pi \int_0^{\xi_0} \left( \frac{\xi^2}{2} - \frac{\xi^6}{6\xi_0^6} \right) \, d\xi
\]

\[
= 2\pi \rho u_{\text{max}} \left[ \frac{\xi_0^2}{2} - \frac{\xi_0^6}{6\xi_0^6} \right] \bigg|_0^{\xi_0}
= 2\pi \rho u_{\text{max}} \left[ \frac{1}{2} - \frac{1}{6} \right]
\]

\[
\dot{m} = \frac{2}{3} \pi \rho u_{\text{max}} \xi_0^2 = \frac{2}{3} \pi \times 999 \times 100 \times 0.25^2
\]

\( \Rightarrow \dot{m} = 13072 \text{ kg/s} \)  

(\text{Answer})

\( \text{Manometer gives:} \)

\[
P_i + \gamma_w h = P_e + \gamma_m h
\]

\( \Rightarrow \)

\[
P_i - P_e = (\gamma_m - \gamma_w) h
\]

\[
h = \frac{P_i - P_e}{\gamma_m - \gamma_w}
\]

Apply momentum eqn:

\[
\int (\rho u_n \xi) \, dA = -\int \rho dA + F_{\text{visc.}} = -\left[ -P_i A + P_e A \right] - 2\pi n \eta_0 L
\]

(\(0, \text{constante}\), \(\text{dev. flow}\))

\(0 = (P_i - P_e) \pi \xi_0^2 - 2\pi n \eta_0 L.\)
\[ -(p_i - p_e) \pi h_0^2 = - \frac{\pi}{2} \frac{2 \pi L h_0}{\pi h_0^2} = \frac{22L}{h_0} \]

and

\[ \frac{d}{du} \mid_{z_2} = \frac{4}{z_2} \frac{d}{d\rho} \mid_{\rho_2} = \frac{4}{z_2} \frac{\mu U_{\text{max}}}{\rho_2} \]

\[ \therefore \ p_i - p_e = \frac{22L}{h_0} = \frac{8}{\mu U_{\text{max}} L} = \frac{8 \times 1.12 \times 10^{-3} \times 100 \times 500}{0.85^2} = 7168 \]

Substitute this in. (\(\text{I}\))

\[ h = \frac{p_i - p_e}{\gamma_m - \gamma_w} = \frac{7168}{9.6 \times 10^3 - 9.8 \times 10^3} = 0.702 \text{ m} \]

\[ \text{C)} \quad \text{Momentum eqn:} \]

\[ \frac{d}{ds} \left( \rho v \cdot n dA \right) = - \int p dA - R_x \]

\[ 0 = - \left[ -(p_i - p_e) \pi h_0^2 - R_x \right] \]

\[ \Rightarrow R_x = (p_i - p_e) \pi h_0^2 = 7168 \times \pi \times 0.25^2 = 1407 \text{ N} \]

\[ \text{D)} \quad \text{Energy eqn:} \]

\[ \dot{v}_i - \dot{v}_e + \frac{\dot{v}_i - \dot{v}_e}{\rho} + \frac{v_i^2}{2} - \frac{v_e^2}{2} + g \left( z_2 - z_1 \right) = \frac{\rho}{\rho} \]

\[ \Rightarrow \dot{v}_i - \dot{v}_e = \frac{p_e - p_i}{\rho} \]

\[ \dot{v}_e - \dot{v}_i = \frac{p_i - p_e}{\rho} \]
\[ C_v(T_e - T_i) = \frac{b_i - b_e}{P_e} \]

\[ C_v \, dT = \frac{7168}{999} = 7.175 \]

\[ dT = \frac{7.175}{4.187 \times 10^3} = 1.71 \times 10^{-3} \text{ K} \]

\[ R_e = \frac{P \, V \, L}{\mu} = \frac{\rho \, m \, \dot{z}_0}{\rho \, A \cdot \mu} = \frac{\dot{m} \, z_0}{\mu A} \]

\[ = \frac{13077 \times 0.25}{1.12 \times 10^{-3} \times 71 \times 0.25^2} = 1.48 \times 10^6 \]

\[ \text{Inviscid means } \mu = 0. \]

\[ \text{a) is unaffected} \]

\[ \text{b) since } h = \frac{b_i - b_e}{\gamma_m - \gamma_w} \text{ and } b_i - b_e = 8 \mu / \sqrt{u_{\text{max}} L} \]

\[ \Rightarrow \quad h = 0 \]

\[ \text{c) Similarly since } R_e = \frac{b_i - b_e}{\gamma_{m} \rho} \]

\[ \Rightarrow \quad \Delta T = \frac{b_i - b_e}{C_v \rho} = 0 \]

\[ \Rightarrow \quad R_e = \frac{P \, V \, L}{\mu} = \infty \]
#5 Inviscid flow throughout
Bernoulli eqn.
\[ p_1 + \frac{1}{2} \rho V_1^2 + \gamma z_1 = p_2 + \frac{1}{2} \rho V_2^2 + \gamma z_2 \]
\[ \gamma (z_1 - z_2) = \frac{1}{2} \rho V_2^2 \]
\[ \gamma h = \frac{1}{2} \rho V_2^2 \]

\[ V_2 = \frac{Q}{\pi R^2} \]

Therefore \[ h = \frac{1}{\gamma} \rho \left( \frac{Q}{\pi R^2} \right)^2 = \frac{p_0^2}{2 \gamma \pi^2 R^4} \] \[ \text{Ans} \]

6 Flow in the nozzle is laminar \( \Rightarrow \) viscous

Therefore, we can't apply Bernoulli eqn between 2 and 3

So inside the nozzle we apply Poiseuille's relation:
\[ Q = \frac{\pi R^4 \Delta p}{8 \mu L} = \frac{\pi R^4 (p_2 - p_3)}{8 \mu L} = \frac{\pi R^4}{8 \mu L} (p_2 - p_{\text{atm}}) \]

Apply Bernoulli's relation between 1 and 2
\[ p_1 + \frac{1}{2} \rho V_1^2 + \gamma z_1 = p_2 + \frac{1}{2} \rho V_2^2 + \gamma z_2 \]
\[ p_{\text{atm}} + \gamma (z_1 - z_2) - \frac{1}{2} \rho V_2^2 = p_2 \]
\[ \Rightarrow p_2 = p_{\text{atm}} + \gamma h - \frac{1}{2} \rho V_2^2 \]
sub. this in \( \text{II} \)
\[ Q = \frac{\pi R^4}{8 \mu L} \left[ p_{\text{atm}} + \gamma h - \frac{1}{2} \rho V_2^2 - p_{\text{atm}} \right] = \frac{\pi R^4}{8 \mu L} \left[ \gamma h - \frac{1}{2} \rho V_2^2 \right] \]
\[ 8 \mu L Q = \pi R^4 (\gamma h - \frac{1}{2} \rho V_2^2) \]
\[ \Rightarrow h = \frac{1}{\gamma} \left[ \frac{8 \mu L Q + \rho Q^2}{\pi R^4} \right] = \frac{8 \mu L Q + Q^2}{\pi R^4 \cdot 2 \gamma \pi^2 R^4} \] \[ \text{Ans} \]