ANALYTICAL SOLUTION FOR DIFFUSION AND CIRCULATION IN A VAPORIZING DROPLET

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The problem of convective diffusion in a liquid droplet with internal circulation is studied. Both the vaporizing and the non-vaporizing cases are considered. An approach which simplifies the governing equations is taken and series solution for both the temperature boundary condition and temperature gradient boundary condition cases are obtained. The results are in reasonably close agreement with the finite-difference solution of the more exact analysis. In the non-vaporizing droplet case, the coupled gas-phase and liquid-phase problem is examined. Solution for the droplet surface temperature in the form of a linear Volterra integral equation, utilizing the series solution, is obtained. The results are very promising for future simplifications in spray combustion analyses.

I. Introduction

Studies on spray vaporization and combustion are of primary importance in analyzing and improving the performance of engines using spray injections. The droplets tend to maintain a relative velocity with the gas in the combustor; therefore, the convective heat and mass transfer associated with the motion are critical and complicate any analysis by introducing multidimensionality to the problem.

The theoretical prediction of the vaporization rate with liquid phase convective motion, involves the solution of the coupled equations of motion, energy and concentration in the gas and liquid phases. This coupling occurs at the gas-liquid interface. The liquid-phase motion and the droplet heating were analyzed by Prakash and Sirignano [1]. In their analysis, the liquid-phase problem was uncoupled from the gas phase by specifying the necessary interface condition. The droplet heating is unsteady and temperature distribution is non-uniform for most of the droplet lifetime under the condition studied. They later developed a gas-phase boundary layer analysis [2] and coupled it to the previous liquid-phase analysis which was modified to account for the changing droplet size due to vaporization. Lara-Urbaneja and Sirignano [3], following the same arguments used in [1] and [2], extended the analysis to the multicomponent droplet case. It was found that the transient behavior prevails throughout the lifetime of the droplet for both heat and mass transfer, for the conditions analyzed. The problem was solved by employing the finite difference scheme in conjunction with matching the gas-phase and liquid-phase solution at the interface. It was realized that [2,3] the algorithm for the droplet vaporization is still too cumbersome for inclusion in a complete combustion analysis and some reasonable analytical simplifications, taking into account the important physics, are mandatory.

The present paper focuses on the problem of convective diffusion in a liquid droplet and strives towards obtaining a simpler solution which requires less computation.

In the uncoupled liquid phase analysis, it was shown [1] that the liquid motion is quasi-steady and consists of a Hill's vortex in the droplet core with a thin viscous boundary layer near the droplet surface and an inviscid internal wake near the axis of symmetry. Similarly, there is a thin thermal boundary layer near the droplet surface and there is coupling to the thermal core in the matching region. The thermal boundary layer which allows for the heat flux to adjust from two-dimensional behavior along the droplet surface to one-dimensional behavior along the boundary layer and thermal core matching region was treated as quasi-steady. A diagram which shows the Hill's vortex within the droplet and identifies the orthogonal streamline coordinates is presented in Figure 1.

Recently, the importance of the thermal boundary layer and the quasi-steadiness assumption have been reviewed by the authors and will appear in a forthcoming publication. The results of the study show that, unless the thermal boundary layer is very thin (very large Peclet numbers), the thermal inertia term is important and the quasi-steadiness assumption is invalid. The results also tend to suggest that the elimination of the thermal boundary
layer and the assumption that the core solution is valid up to the droplet surface may still give solutions with acceptable degree of accuracy. Note that while the thermal boundary layer formulation is two-dimensional, the thermal core formulation is one-dimensional. Therefore, the removal of the thermal boundary layer will simplify the problem tremendously. With that simplification in mind, the core solution is examined in the present study. Simpler solutions for both the vaporizing and the non-vaporizing cases are obtained. The results are presented in the following section.

II. Thermal Core Solution

In Prakash and Sirignano’s analysis [1,2], the droplet core heating was shown to be essentially unsteady and normal to the closed streamlines because of the very short residence and circulation time along a streamline compared to the droplet lifetime. Therefore, the orthogonal streamline coordinates were used for the energy equation in the droplet core. In the orthogonal streamline coordinate system ($\xi$, $\psi$, $\eta$), the scale factors are given by

\[ h_\xi = \frac{AR^2}{8} \int \frac{h_\psi h_\eta}{h_\xi} d\xi \]  \hspace{1cm} (3) \\
\[ g_2 (\phi) = \frac{8}{AR^3} \int \frac{h_\psi h_\eta}{h_\phi} d\xi \]  \hspace{1cm} (4)

The energy equation valid in the core was found to be

\[ \frac{\partial T}{\partial \tau'} = a (\tau', \phi) \frac{\partial^2 T}{\partial \phi^2} + b (\tau', \phi) \frac{\partial T}{\partial \phi} \]  \hspace{1cm} (5)

where

\[ a (\tau', \phi) = \frac{R_0^2 g_2 (\phi)}{g_1 (\phi)} \]  \hspace{1cm} (6) \\
\[ b (\tau', \phi) = \frac{1}{g_1 (\phi)} R^2 \frac{dg_2 (\phi)}{d\phi} \]

\[ + \frac{3}{g_1 (\phi) R} \int_0^\phi \frac{g_1 (\phi') d\phi'}{R} \]  \hspace{1cm} (7)

\[ T = \frac{T_{w} - T_0}{T_b - T_0} \]  \hspace{1cm} (8)

\[ \tau' = \alpha t R_0^2 \]  \hspace{1cm} (9)

The initial and boundary conditions were:

(i) at $\tau' = 0$, $T = 0$

(ii) at $\phi = 0$, $\partial T / \partial \tau' = b (\tau', \phi) \partial T / \partial \phi$

(iii) at $\phi = \phi_0$, $\partial T / \partial \phi = q_1 (\tau')$

The boundary $\phi = \phi_0$ corresponds to the outer boundary of the core near the droplet surface. In the boundary condition (iii) above, the temperature gradient $q_1 (\tau')$ was obtained by requiring the continuity of heat flux in the interfacial region of the thermal boundary layer and the thermal core. Equation (5) was solved numerically using a Crank-Nicholson scheme and the solution was marched forward in time.

To seek a simpler solution to Equation (5), the functions $g_1 (\phi)$ and $g_2 (\phi)$ are examined in this study. They are evaluated numerically and are shown graphically in Figure 2. In most regions, the function $g_1 (\phi)$ varies slowly with $\phi$ while the function $g_2 (\phi)$ varies linearly with $\phi$. Approximating them by

\[ g_1 (\phi) = b_0; g_2 (\phi) = b_1 \phi \]  \hspace{1cm} (11)

where $b_0$ and $b_1$ are constants, Equation (5) simplifies to

\[ g_1 (\phi) = \frac{AR^2}{8} \int \frac{h_\psi h_\eta}{h_\xi} d\xi \]  \hspace{1cm} (3) \\
\[ g_2 (\phi) = \frac{8}{AR^3} \int \frac{h_\psi h_\eta}{h_\phi} d\xi \]  \hspace{1cm} (4)

\[ \frac{\partial T}{\partial \tau'} = a (\tau', \phi) \frac{\partial^2 T}{\partial \phi^2} + b (\tau', \phi) \frac{\partial T}{\partial \phi} \]  \hspace{1cm} (5)

where

\[ a (\tau', \phi) = \frac{R_0^2 g_2 (\phi)}{g_1 (\phi)} \]  \hspace{1cm} (6) \\
\[ b (\tau', \phi) = \frac{1}{g_1 (\phi)} R^2 \frac{dg_2 (\phi)}{d\phi} \]

\[ + \frac{3}{g_1 (\phi) R} \int_0^\phi \frac{g_1 (\phi') d\phi'}{R} \]  \hspace{1cm} (7)

\[ T = \frac{T_{w} - T_0}{T_b - T_0} \]  \hspace{1cm} (8)

\[ \tau' = \alpha t R_0^2 \]  \hspace{1cm} (9)

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\[ g_1 (\phi) = b_0; g_2 (\phi) = b_1 \phi \]  \hspace{1cm} (11)

where $b_0$ and $b_1$ are constants, Equation (5) simplifies to
\[
\frac{\partial T}{\partial \tau} = \phi \frac{\partial^2 T}{\partial \phi^2} + \left[ 1 + C(\tau) \phi \right] \frac{\partial T}{\partial \phi} \tag{12}
\]

where

\[
d\tau = \frac{b_1 R_0^2}{b_0 R^2} d\tau' \tag{13}
\]

and

\[
C(\tau) = 2 \left( \frac{R_0}{R} \right)^{3/2} \frac{d}{d\tau} \left( \frac{R}{R_0} \right)^{3/2} \tag{14}
\]

The function \( C(\tau) \) relates to the droplet vaporization rate and is coupled to the gas-phase solution. In general, \( C(\tau) \) would be a function of \( \tau \). Nevertheless, the simple case where \( C(\tau) \) equals a constant \( C_0 \), is examined first, followed by the more general time-varying \( C \) case. For \( C = C_0 \), Equation (12) and the associated initial and boundary conditions (10) become

\[
\frac{\partial T}{\partial \tau} = \phi \frac{\partial^2 T}{\partial \phi^2} + (1 + C_0 \phi) \frac{\partial T}{\partial \phi} \tag{15}
\]

and

\[
F(s, C_0, \phi) = 1 + \sum_{k=1}^{\infty} \frac{s(s - C_0) \cdots [s - (k - 1) C_0]}{(k)!^2} \phi^k \tag{20}
\]
and
\[ G(t, C_0, \phi) = \frac{\partial F}{\partial \phi} (t, C_0, \phi) \]  
(21)

and by means of the Inversion Integral Technique
\[ T(t, \phi) = \frac{1}{\beta(t)} \left( 1 + \sum_{n=1}^{\infty} \frac{F(\lambda_n, C_0, \phi)}{\lambda_n^2} e^{\lambda_n^2 t/4 \phi_0} \right) \]  
(22)

where the \( \lambda_n \)'s are the roots of \( G(\lambda_n, C_0, \phi_0) = 0 \).

In (22),
\[ F(t, C_0, \phi) = \frac{\partial}{\partial t} F(t, C_0, \phi) \bigg|_{t=0} \]  
(23)
\[ G(t, C_0, \phi) = \frac{\partial}{\partial t} G(t, C_0, \phi) \bigg|_{t=0} \]  
(24)
\[ C(t, C_0, \phi) = \frac{\partial^2}{\partial t^2} G(t, C_0, \phi) \bigg|_{t=0} \]  
(25)

For the unity temperature case, i.e. \( T(t, \phi) = 1 \), it can be shown that the temperature distribution is given by
\[ T(t, \phi) = 1 + \sum_{n=1}^{\infty} \frac{F(\lambda_n, C_0, \phi_0)}{\lambda_n^2} e^{\lambda_n^2 t/4 \phi_0} \]  
(26)

where the \( \lambda_n \)'s are the roots of \( F(\lambda_n, C_0, \phi_0) = 0 \).

For the non-vaporizing case, \( C_0 = 0 \), the solutions can be expressed in terms of Bessel functions. For the unity temperature gradient case,
\[ T(t, \phi) = \phi - \frac{\phi_0}{2} + \tau \]  
- \[ 4 \phi \sum_{n=1}^{\infty} \frac{J_0(\lambda_n \sqrt{\phi/\phi_0}) e^{-\lambda_n^2 t/4 \phi_0}}{\lambda_n^2 J_0(\lambda_n)} \]  
(27)

where \( J_1(\lambda_n) = 0 \). For the unity temperature case,
\[ T(t, \phi) = 1 \]  
- \[ 2 \frac{\sum_{n=1}^{\infty} J_0(\lambda_n \sqrt{\phi/\phi_0}) e^{-\lambda_n^2 t/4 \phi_0}}{J_1(\lambda_n)} \]  
(28)

where \( J_0(\lambda_n) = 0 \).

For the time dependent \( C \) case, \( C = C(\tau) \); by the following change of variables:
\[ V(\tau, \phi) = T(\tau, \phi) \sech \left( \frac{k \tau}{2} \right) e^{-k \phi} \]  
(29)

Equations (12) and (16) are transformed to
\[ \frac{\partial V}{\partial \tau} = \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial V}{\partial \phi} - \frac{\phi V}{4} \]  
(30)

(i) at \( \tau = 0, V = 0 \)
(ii) at \( \phi = 0, \frac{\partial V}{\partial \phi} = \frac{\partial V}{\partial \tau} \)
(iii) at \( \phi = \phi_0, \frac{\partial V}{\partial \phi} = q_2(\tau) \sech \left( k \tau/2 \right) e^{-C \phi_0^2/2} \)

where \( C(\tau) \) in this case is approximated by
\[ C = -k \tanh \left( k \tau/2 \right) \]  
(32)

and the constant \( k \) is determined such that Equation (32) provides the best fit to a given function \( C \). Later, we will demonstrate that acceptable fits can be established. The solution for \( V \) is given by
\[ V(\tau, \phi) = u(0, \phi) \left[ f(\tau) + \frac{1}{2} C(\tau) V(\tau, \phi_0) \right] \]  
+ \[ \int_0^\tau \frac{\partial u(t, \tau, \phi_0)}{\partial \tau} (\tau - t, \phi_0) dx \]  
\[ + \frac{1}{2} \int_0^\tau C(\tau) V(\tau, \phi_0) \frac{\partial u(t, \tau, \phi_0)}{\partial \tau} (\tau - t, \phi_0) dx \]  
(33)

and
\[ u(\tau, \phi) = \frac{W(0, k, \phi)}{W_0(0, k, \phi_0)} \]  
+ \[ \sum_{n=1}^{\infty} \frac{W(\lambda_n, k, \phi_0) e^{\lambda_n \phi_0}}{\lambda_n W_{\phi_0}(\lambda_n, k, \phi_0)} \]  
(34)

where the \( \lambda_n \)'s are the roots of \( W_0(\lambda_n, k, \phi_0) = 0 \).

In equations (33) and (34) above,
\[ W(t, k, \phi) = \sum_{n=0}^{\infty} \lambda_n \phi^n \]  
(35)

where \( A_0 = 1, A_1 = \tau, A_{n+2} = (e A_{n+1} + k^2 A_n)/4(n + 2)^2 \) for \( n \geq 0 \).

\[ W_{\phi_0}(\lambda_n, k, \phi_0) = \frac{\partial W(t, k, \phi_0)}{\partial \phi} \bigg|_{t=0} \]  
(36)

\[ W_{\phi_0}(\lambda_n, k, \phi_0) = \frac{\partial^2 W(t, k, \phi_0)}{\partial \phi^2} \bigg|_{t=0} \]  
(37)
and
\[
J(t) = q_s(t) \frac{1}{\tanh(k\tau/2)} e^{-k \phi_0 / 2} \tanh(k\tau/2) \quad (38)
\]

For the temperature boundary condition case, i.e.,
\[ T(t, \phi) = T(t) \text{,} \]
the solution is given by
\[
V(t, \phi) = \omega(t, \phi) g(0) + \int_0^\infty \frac{d\omega(x)}{dx} \ dx \quad (39)
\]

\[
\omega(t, \phi) = \frac{W(0, k, \phi)}{W(0, k, \phi_0)} + \sum_{n=1}^\infty \lambda_n W_n(\lambda_n, k, \phi_0) e^{\lambda_n t} \quad (40)
\]

where the \( \lambda_n \)'s are the roots of \( W(\lambda_n, k, \phi_0) = 0 \).
In equation (39),
\[
h(t) = \frac{g(t)}{\tanh(k\tau/2)} e^{C o o v_2} \quad (41)
\]

III. Results and Discussion

For the non-vaporizing droplet case, the series solutions given by Equations (27) and (28) were calculated and the results are shown in Figures 3 and 4. The finite difference solutions for the unsimplified equation [i.e., Equation (5)], were also calculated and are plotted alongside those mentioned previously. The results are in close agreement. The value of \( \phi_0 \) was chosen as 0.99 in all the calculations. Equation (13) was also solved by the same finite-difference scheme employed for Equation (5) and its solution was identical to the analytical solution, guaranteeing that the finite-difference scheme used does give exact solution. Therefore, the differences in Figures 3 and 4 result solely from the approximations for \( g_1 \) and \( g_2 \) given before. Truncation errors and round-off errors contribute negligibly.

In the constant C vaporizing droplet case, a representative value of \( C_0 \) was obtained after a brief review of the results obtained in [2]. The series solutions given by Equations (22) and (26) were then computed and the results for the unity temperature gradient and the unity temperature boundary conditions are shown in Figures 5 and 6, respectively. As in the non-vaporizing case, the finite-difference solutions for the unsimplified equations [i.e., Equations (5)] were also calculated and are also plotted alongside the corresponding series solutions. The value of \( C_0 \) was chosen to be -0.525. The results for both the unity temperature gradient boundary condition case and the unity temperature case are in reasonably close agreement.

As mentioned earlier, the differences in the figures result solely from the approximations for \( g_1 \) and \( g_2 \). A closer agreement between the results of the series solutions of the approximated equation and the finite difference solutions of the unsimplified equation can therefore be expected by improving the approximation for \( g_1 \). It was found that for both the vaporizing droplet case and the non-vaporizing droplet case, series solutions can still be obtained for a polynomial approximation of \( g_1 \). A first or second order polynomial approximates \( g_1 \) reasonably well and the results were found to be in very good agreement with the finite difference solution. It is obvious that the series solution scheme consumes much less computation time than the finite difference method.

It should be mentioned that the series solutions for both the unity temperature and the unity temperature gradient boundary condition cases can be used to generate solutions for the corresponding time dependent boundary condition cases. In [1], it was shown that the mathematical equations and procedure for the thermal core are similar to those used by Brignell [4] for the problem of solute extraction from an internally circulating liquid drop. On the arguments that the boundary layer is thin and that the diffusion in the core is the rate controlling process, Brignell could solve the problem of mass transfer from the droplet by neglecting the existence of the boundary layer and solving the diffusion equation for the core. He imposed the droplet surface concentration as the boundary condition for the outer boundary of the core, thus eliminating the coupling between the boundary layer and the core without causing any significant errors in the mass transfer problem. The series solution obtained in the section for the non-vaporizing droplet case with unity temperature boundary condition is therefore applicable to Brignell's solute extraction problem. The fact that series solution exists for any polynomial approximation of \( g_1 \) allows the series solution [i.e., Equation (26)] to provide a quick and easy way of obtaining high accuracy approximate solution for the solute extraction problem.

For the time varying C vaporizing droplet case, C is approximated by some exponential functions given by Equation (32). The values of C as a function of time obtained from the results in [3] are shown in Figure 7 along with the exponential approximations. As indicated in the figure, a reasonable curve fit by Equation (32) can indeed be found. The series solution given by Equation (39) was calculated for the case of unity temperature boundary condition with \( k = 1.40 \). The results shown in Figure 8 are again in reasonable agreement with the finite difference solution of Equation (5).

It is of interest to mention that the series solutions given by (22), (26), (27) or (29) for the constant C case require about 4 terms for convergence to the desired accuracy and on the average con-
Fig. 3. Temperature versus \( \phi \) comparisons at various time instants with unity temperature gradient boundary condition and \( C = 0 \).
Fig. 4. Temperature versus $\phi$ comparisons at various time instants with unity temperature boundary condition and $C = 0$. 

DIFFUSION AND CIRCULATION IN VAPORIZING DROPLET
Fig. 5. Temperature versus $\phi$ comparisons at various time instants with unity temperature gradient boundary condition and $C = -0.628$. 
Fig. 6. Temperature versus $\phi$ comparisons at various time instants with unity temperature boundary condition and $C = -0.828$. 
DROPLET COMBUSTION

requires matching the temperature at the gas-liquid interface. In particular, the coupled gas phase and liquid phase problems must be considered here. However, it may be expected that only the temperature at the outer core boundary is needed for determining the vaporization rate. Actually, we need not know the temperature in the droplet interior to determine the heat flux to the droplet and its vaporization rate. This will reduce the amount of computation required for the integral equation tremendously. That is, the one-dimensional, unsteady partial differential equation in the liquid core could be reduced to a time dependent Volterra Equation, thereby eliminating spatial dependence from the problem. In the appendix, the application of this approach to the case of a non-vaporizing droplet is given. The results, shown in Figure 9, are in surprisingly good agreement with the more exact finite-difference solution. The application of this approach to the vaporizing droplet case will be presented in a future publication. Nevertheless, it is believed that the solutions obtained for the varying C vaporizing droplet case may have potential value in constructing simpler solutions for the overall droplet vaporizing problem.

IV. Conclusions

The problem of convective diffusion in liquid droplet with internal circulation has been analyzed for both the vaporizing and the non-vaporizing cases. Based on the results of a recent study, the liquid thermal boundary layer is eliminated and the core solution is considered to be valid up to the droplet surface. This removal of the thermal boundary layer simplifies the problem from a two-dimensional situation to a one-dimensional case. The energy equation for the thermal core is further simplified and series solutions for both the constant temperature boundary condition and the constant temperature gradient boundary condition cases are obtained. The results are in reasonably close agreement with the finite-difference solutions of the unsimplified equation. Because of the linearity nature of the problem, the series solutions can be superimposed to generate solutions for the time varying boundary condition cases. In the non-vaporizing case, the coupled gas-phase and liquid-phase problem is examined. With the legitimate assumption of constant Nusselt number and utilizing the series solution, solution for the droplet surface temperature in the form of a linear Volterra type integral is obtained. Within the acceptable range of accuracy, the integral equation approach appears to be more efficient than the finite difference solution scheme.

The results are most promising for treating the spray combustion problem in a more simplified and manageable manner since the droplet heating prob-

—CPU time on a DEC-20 computer system.

FIG. 7. C(t) versus t.

sume about 10 microseconds of CPU* time. The series solution given by Equation (39) for the time varying C case requires about 6 terms for convergence to the desired accuracy and on the average consumes about 30 microseconds of CPU time. Conversely, the finite difference solutions require about 50 microseconds of CPU time for each time step, thus each solution requires about 5 minutes of CPU time for the time interval of interest and for the same accuracy as the series solutions.

Attempts were made to extend the constant k series solution to the time varying k case. The solution at any given time is obtained by treating k as constant and computing the corresponding series solution of the constant k case. The "constant" value of k is set equal to the value of k at which the approximation curve intersects the exact C curve at that time instant. The results by using this time dependent k scheme were found to be in acceptable agreement with the finite difference solution. Similar attempts were also made to extend the series solutions of the constant C vaporizing droplet case to the varying C vaporizing droplet case. Unfortunately, the results did not compare well with the finite difference solution (after the initial diffusion period) unless C is varying very slowly.

The solution for the time dependent temperature gradient boundary condition case [i.e., Eq. (33)] involves Volterra type integral equation with series kernel and appears to be lengthy and requires considerable computation in order to obtain the temperature distribution within the core. But, as mentioned earlier, the final goal is to determine the droplet heating rate and vaporization rate and this
Fig. 8. Temperature versus $\phi$ comparisons at various time instants with unity temperature boundary condition and time varying $C$. 

Δ EXACT: $C = C(\tau)$  
+: APPROXIMATE: $C = \frac{\tau}{K} \tanh(K\tau/2)$  
$K = 1.20$  
$\tau = 0.10$  
$\tau = 0.50$  
$\tau = 1.00$
lem is reduced from a multi-dimensional time-dependent analysis for the liquid interior temperature to only a time-dependent analysis for the surface temperature.

APPENDIX: Coupled Gas-Liquid Solution in the Non-vaporizing Case

In the non-vaporizing case, the Nusselt number is a function of Reynolds number, but independent of temperature so that, for our purposes, it may be considered to be constant with time. It can be shown that the temperature gradient at the droplet surface is given by

$$\frac{dT}{d\phi} = A + B T_s$$ \hspace{1cm} (A-1)

where

$$A = \frac{Nu}{8} \left( \frac{k_x}{k_\ell} \right) T_s$$

and

$$B = -\frac{Nu}{8} \left( \frac{k_x}{k_\ell} \right)$$

The solution for the non-vaporizing droplet with time-varying temperature gradient boundary condition $\partial T/\partial \phi = f(\tau)$ is given by

$$T(\tau, \phi) = \left[ \phi - \frac{1}{2} - 4 \sum_{n=1}^{\infty} \frac{J_0(\lambda_n \sqrt{\phi})}{\lambda_n^2 J_0(\lambda_n)} \right] f(\tau)$$

$$+ \int_0^\tau f(x) \, dx$$

$$+ \sum_{n=1}^{\infty} \frac{J_0(\lambda_n \sqrt{\phi})}{J_0(\lambda_n)} \int_0^\tau f(x) e^{-\lambda_n \kappa (\tau-x)/4} \, dx$$ \hspace{1cm} (A-2)

Substituting (A-1) for $f(\tau)$ and evaluating at $\phi = 1$ the surface temperature after some algebraic manipulation is governed by

$$T_s(\tau) = g(\tau) + \int_0^\tau k(\tau-x) T_s(x) \, dx \hspace{1cm} (A-3)$$

which is a linear Volterra Integral Equation of the second kind.

where

$$g(\tau) = A \left[ \frac{\tau^2}{2} - 4 \sum_{n=1}^{\infty} \frac{e^{-\lambda_n \kappa /4}}{\lambda_n^2} \right]$$

and

$$k(\tau-x) = B \left[ 1 + \sum_{n=1}^{\infty} e^{-\lambda_n \kappa (\tau-x)/4} \right]$$

The solution of the integral equation (A-3) can be obtained readily by means of product integration method as suggested in (5) and (6). The surface temperature $T_s$ can be generated from the following recurrence formula

$$g(nh) + \frac{\sum_{k=1}^{n-1} A(k)T([n-k]+1)}{1-i_1(1)} \hspace{1cm} n > 1$$

where

$$A(k) = i_0(k) + i_1(k+1) - i_1(k);$$

$$i_0(k) = B \left[ h + 4 \left( S_1(k-1) - S_2(k) \right) \right];$$

$$i_1(k) = B \left[ \frac{h}{2} + 4 S_1(k-1) \right.$$

$$\left. \frac{4}{h} \left( S_2(k-1) - S_2(k) \right) \right]$$ \hspace{1cm} (A-5)

The result for surface temperature in a case when $A = 2.4842$ and $B = -0.4333$ is presented in Figure 9. As expected, surface temperature increases with time due to droplet heating. The comparison with a finite-difference calculation using exact $g_1$ and $g_2$ values is shown in the same figure to be very favorable. A finite-difference calculation employing the approximate $g_1$ and $g_2$ values resulted in differences with the integral equation solution that could not be detected graphically. Once the surface temperature is known, Equation (A-3) may be utilized to determine the gradient at the surface and, therefore, the heat flux as functions of time. Now, (A-2) would yield the behavior of the temperature in the droplet interior. The CPU time for the finite difference solution and the integral equation solution are respectively 0.80 and 0.15 seconds for 1st K accuracy. The integral equation approach
Fig. 9. Surface temperature versus time comparisons.
appears to be more efficient. But the efficiency will
decrease as accuracy increases.

Nomenclature

\( h_x \) scale factor in the streamwise direction
\( h_n \) scale factor in the direction normal to the streamline
\( h_i \) scale factor in the azimuthal direction
\( i \) \( \sqrt{-1} \)
\( J_0 \) Bessel function of the first kind and order zero
\( J_1 \) Bessel function of the first kind and order one
\( k_x, k_i \) thermal conductivity in the gas phase and liquid phase, respectively
\( Nu \) Nusselt number
\( p \) \( r/R \)
\( R \) droplet radius
\( R_0 \) initial droplet radius
\( s \) Laplace Transform parameter
\( t \) physical time
\( T \) non-dimensional temperature
\( \dot{T} \) non-dimensional temperature \( T \) after the Laplace transform on time
\( T_w \) average streamline temperature
\( T_b \) boiling temperature
\( T_0 \) initial temperature
\( T_e \) non-dimensional droplet surface temperature
\( T_a \) ambient temperature
\( U \) velocity in the streamwise direction
\( \alpha_f \) fluid thermal diffusivity
\( \tau_1, \tau \) non-dimensional times defined by (9) and (13), respectively
\( \xi \) streamwise coordinate
\( \phi \) dimensionless stream function
\( \theta \) tangential coordinate
\( \lambda_n \) eigenvalues for series solutions

(Other symbols are defined in the text).

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REFERENCES


COMMENTS

Prof. Ken K. Kuo, The Pennsylvania State University, University Park, PA. I must say that your paper contains a very elegant analysis and a reasonable simplification procedure. However, I believe that even your simplified analysis is still too cumbersome to be coupled with any comprehensive spray model. Would you please comment on an alternative approach, for example, the use of the more exact analysis, developed earlier by Prakash and Sirignano (1) in a parametric study to generate a suitable correlation between burning rate, pressure, ambient oxidizer concentration, internal circulation, etc? Such a correlation (after being verified experimentally) could readily be incorporated into a spray model.

REFERENCE


Author’s Reply. It seems to us that the approach suggested by Professor Kuo will be a possible alternative useful for comprehensive spray models. However, the fact that the problem is basically transient in nature may introduce complexities to the correlation scheme. Someone should be encouraged to do it.