CHAPTER VI: DIMENSIONAL ANALYSIS

6.1 Introduction: What is Dimensional Analysis?

Basically, dimensional analysis is a method for reducing the number and complexity of experimental variables which affect a given physical phenomenon using a compacting technique. If a phenomenon depends upon \( n \) dimensional variables, dimensional analysis will reduce the problem to only \( n-k \) dimensionless variables where the reduction \( k \) will depend on the problem complexity. Generally \( k \) equals the number of different dimensions which govern the problem. In fluid mechanics the three basic dimensions are usually taken to be mass, \( m \); length, \( L \); and time, \( t \).

Suppose the force, \( F \), acting on a particular body immersed in a fluid flow depends on the body length, \( L \), flow velocity, \( u \), fluid density, \( \rho \), and fluid viscosity, \( \mu \); that is

\[
F = f(L, u, \rho, \mu)
\]

Generally speaking, it takes about 10 experimental points to define a curve. In order to find the affect of \( L, u, \rho, \) and \( \mu \) on \( F \), we have to do \( 10^4 \) experiments. With dimensional analysis, as it will be shown later that

\[
\frac{F}{\rho u^2 L^2} = g \left( \frac{\rho u L}{\mu} \right) / C_F \quad \text{Re} \quad \text{Force Coefficient} \quad \text{Reynolds Number}
\]

we can establish the function, \( g \), by running the experiment for only 10 values of \( \text{Re} \). We do not have to vary \( u, L, \rho, \) or \( \mu \) separately but only the grouping \( \rho u L / \mu \).
6.2 Buckingham's \( \Pi \) Theorem

If an equation truly expresses a proper relationship between variables in a physical process, it will be dimensionally homogeneous; that is, each of its additive terms will have the same dimensions. For example:

\[
P_1 \frac{1}{\rho} + \frac{V_1^2}{2} + gZ_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gZ_2
\]

each term has the same dimension, \([L^2/T^2]\).

If it is known that a physical process is governed by a dimensionally homogeneous relation involving \( n \) dimensional parameters, such as

\[
X_1 = f(X_2, X_3, \ldots, X_n)
\]

where the \( X \)s are dimensional variables, there exists an equivalent relation involving a smaller number, \( n-k \), of dimensionless parameters such that

\[
\Pi_1 = F(\Pi_2, \Pi_3, \ldots, \Pi_{n-k})
\]

where the \( \Pi \)'s are dimensionless groups constructed from the \( X \)s. The reduction, \( k \), is usually equal to (but never more than) the number of fundamental dimensions involved in the \( X \)s.
Method: (Mechanical energy loss in a pipe flow is used as an example for illustration purpose)

1. Write a functional expression for the dimensional relation under investigation. Be sure to include all relevant dimensional parameters such as

$$ gh_L = f(L, D, u, \rho, \mu, \varepsilon) $$

where

- $L$ = pipe length
- $D$ = pipe diameter
- $u$ = flow velocity
- $\rho$ = density of fluid
- $\mu$ = fluid viscosity
- $\varepsilon$ = pipe roughness
- $gh_L$ = energy loss per unit of mass flow

2. Determine the number of dimensionless parameters you need to construct. This number is equal to the number of dimensional parameters, $n$, in the functional relation minus a number, $k$, that is equal to the maximum number of dimensional parameters that cannot form a dimensionless group, $\Pi$, among themselves. This number, $k$, is usually equal to the number of fundamental dimensions involved in the dimensional parameters. It is never greater than the number of fundamental dimensions; that is

$$ n = 7, $$
$$ k = 3, $$
$$ n-k = 7 - 3 $$
$$ n-k = 4 $$

3. Select $k$ dimensional parameters that contain among them all of the fundamental dimensions. Combine these parameters with the remaining $n-k$ dimensional parameters to form the required $n-k$ number of dimensionless parameters. This is done by selecting the remaining $n-k$ parameters one at a time and multiplying by appropriate powers of the $k$ repeating variables so that the result is dimensionless.
For example, select \( p, u, \) and \( D \) as the repeating variables, the remaining four parameters are \( gh_L, L, \mu, \) and \( \varepsilon. \)

Determine the first dimensionless parameter.

\[
\Pi_1 = gh_L \rho^a u^b D^c
\]

Find \( a, b, \) and \( c \) such that \( \Pi_1 \) is dimensionless. Substituting the dimensions of each individual terms gives

\[
\left[ \Pi_1 \right] = \left[ \frac{L^2}{T^2} \right] \left[ \frac{M}{L^3} \right] ^a \left[ \frac{L}{T} \right] ^b \left[ L \right] ^c
\]

Combining exponents we have

\[
\left[ \Pi_1 \right] = [M]^a [L]^{2-3a+b+c} [T]^{-2-b}
\]

In order for \( [\Pi_1] \) to be dimensionless the exponents should all vanish. Therefore

\[
\begin{align*}
   a &= 0 \\
   2 - 3a + b + c &= 0 \\
   -2 - b &= 0
\end{align*}
\]

or

\[
\begin{align*}
   a &= 0 \\
   b &= -2 \\
   c &= 0
\end{align*}
\]

Which gives

\[
\Pi_1 = gh_L \frac{u^2}{\mathcal{U}} = \frac{gh_L}{\mathcal{V}^2_{\mathcal{U}}}
\]
Determine the second dimensionless parameter.

\[ \Pi_2 = L \rho^a u^b D^c \]

\[ [\Pi_2] = [L]\left(\frac{M}{L^3}\right)^a \left(\frac{L}{T}\right)^b [L]^c \]

\[ [\Pi_2] = [M]^a [L]^{-3a+b+c} [T]^b \]

\[ a = 0, \quad b = 0, \quad c = -1 \]

Therefore

\[ \Pi_2 = \frac{L}{D} \]

Determine the third dimensionless parameter.

\[ \Pi_3 = \mu \rho^a u^b D^c \]

\[ [\Pi_3] = \left(\frac{M}{LT}\right) \left(\frac{M}{L^3}\right)^a \left(\frac{L}{T}\right)^b [L]^c \]

\[ [\Pi_3] = [M]^{1+a} [L]^{-1-3a+b+c} [T]^{-1-b} \]

\[ a = -1, \quad b = -1, \quad c = -1 \]

Therefore

\[ \Pi_3 = \frac{\mu}{\rho \cdot u \cdot D} \]
Find the fourth dimensionless parameter

\[ \Pi_4 = \varepsilon \rho^a u^b D^c \]

\[ [\Pi_4] = [L] \left[ \frac{M}{L^3} \right]^a \left[ \frac{L}{T} \right]^{-b} \left[ L \right]^c \]

\[ [\Pi_4] = [M]^a [L]^{1-3a+b+c} [T]^b \]

\[ a = 0, \ b = 0, \ c = -1 \]

Therefore

\[ \Pi_4 = \frac{\varepsilon}{D} \]

The final result can be written as

\[ \frac{ghL}{U^2} = F \left( \frac{L}{D'}, \frac{\mu}{\rho u D'}, \frac{\varepsilon}{D} \right) \]