3.3 Water flows steadily through the variable area horizontal pipe shown in Fig. P3.3. The velocity is given by \( V = 10(1 + x) \) ft/s, where \( x \) is in feet. Viscous effects are neglected. (a) Determine the pressure gradient, \( \frac{dp}{dx} \) (as a function of \( x \)) needed to produce this flow. (b) If the pressure at section (1) is 50 psi, determine the pressure at (2) by (i) integration of the pressure gradient obtained in (a); (ii) application of the Bernoulli equation.

(a) 
\[
-\sin \theta \frac{dv}{ds} = \rho \frac{dV}{ds} \quad \text{but} \quad \theta = 0 \quad \text{and} \quad V = 10(1 + x) \text{ ft/s}
\]

\[
\frac{dp}{dx} = -\rho \frac{dV}{dx} \quad \text{or} \quad \frac{dp}{dx} = -\rho \frac{dv}{ds} \frac{ds}{dx} = -\rho \frac{dv}{dx} = -\rho \left( 10(1 + x) \right)
\]

Thus, \( \frac{dp}{dx} = -194 \frac{\sin \theta}{144} \left( \frac{10 \text{ ft}}{s} \right) (1 + x) \), with \( x \) in feet

\[
= -194(1 + x) \frac{16}{144}
\]

(b)(i) \( \frac{dp}{dx} = -194(1 + x) \) so that

\[
\int_{p_1}^{p_2} dp = -194 \int_{0}^{x_2} (1 + x) \, dx
\]

or \( p_2 = 50 \text{ psi} - 194 \left(3 + \frac{3}{2} \right) \frac{16}{144} \left( \frac{11^2}{144.0 \text{ ft}^2} \right) = 50 - 10.1 = 39.9 \text{ psi} \)

(ii) \( p_1 + \frac{1}{2} \rho V_1^2 + \rho Z_1 = p_2 + \frac{1}{2} \rho V_2^2 + \rho Z_2 \) or with \( Z_1 = Z_2 \)

\[
p_2 = p_1 + \frac{1}{2} \rho (V_2^2 - V_1^2) \quad \text{where} \quad V_1 = 10(1 + 0) = 10 \text{ ft/s}
\]

\[
V_2 = 10(1 + 3) = 40 \text{ ft/s}
\]

Thus,

\[
p_2 = 50 \text{ psi} + \frac{1}{2} \left(194 \frac{\sin \theta}{144} \right) \left(10^2 - 40^2 \right) \frac{11^2}{144.0 \text{ ft}^2} \left( \frac{11^2}{144.0 \text{ ft}^2} \right) = 39.9 \text{ psi}
\]
3.5 An incompressible fluid with density $\rho$ flows steadily past the object shown in Video V3.7 and Fig. P3.5. The fluid velocity along the horizontal dividing streamline ($-\infty \leq x \leq -a$) is found to be $V = V_0 (1 + a/x)$, where $a$ is the radius of curvature of the front of the object and $V_0$ is the upstream velocity. (a) Determine the pressure gradient along this streamline. (b) If the upstream pressure is $p_0$, integrate the pressure gradient to obtain the pressure $p(x)$ for $-\infty \leq x \leq -a$.

(c) Show from the result of part (b) that the pressure at the stagnation point ($x = -a$) is $p_0 + \rho V_0^2/2$, as expected from the Bernoulli equation.

\[ (a) \quad \frac{\partial p}{\partial x} = -\rho V \frac{\partial V}{\partial x}, \quad \text{where} \quad V = V_0 \left(1 + \frac{a}{x}\right) \]

Thus, \[ \frac{\partial V}{\partial x} = \frac{dV}{dx} = -\frac{V_0 a}{x^2}. \]

or \[ \frac{\partial p}{\partial x} = -\rho V_0 \left(1 + \frac{a}{x}\right) \left(-\frac{V_0 a}{x^2}\right) = \rho a V_0^2 \left(\frac{1}{x^2} + \frac{a}{x^3}\right) \]

\[ (b) \quad \left\{ \begin{align*}
\frac{\partial p}{\partial x} &= \int_{-\infty}^{x} \frac{\partial p}{\partial x} \, dx = \rho a V_0^2 \int_{-\infty}^{x} \left(\frac{1}{x^2} + \frac{a}{x^3}\right) \, dx \\
&= \rho a V_0^2 \left[ -\frac{1}{x} - \frac{1}{2} \frac{a}{x^2} \right]_{-\infty}^{x} \\
&= \rho - \rho_0 = \rho a V_0^2 \left[ -\frac{1}{x} - \frac{1}{2} \frac{a}{x^2} \right]_{-\infty}^{x} \\
&= \rho_0 - \rho a V_0^2 \left[ -\frac{1}{x} - \frac{1}{2} \frac{a}{x^2} \right]_{-\infty}^{x} \\
&= \rho_0 - \rho a V_0^2 \left[ -\frac{1}{2} - \frac{1}{2} \frac{a}{x^2} \right] \\
&= \rho_0 + \frac{1}{2} \rho a V_0^2 \\
\end{align*} \]

From part (b), when $x = -a$

\[ \rho \bigg|_{x = -a} = \rho_0 - \rho a V_0^2 \left[ -\frac{1}{2} + \frac{a}{2a^2} \right] = \rho_0 + \frac{1}{2} \rho V_0^2 \\
\]

From the Bernoulli equation \[ \rho_0 + \frac{1}{2} \rho V_0^2 = \rho_1 + \frac{1}{2} \rho V_1^2 \]

where \[ V_1 = V \bigg|_{x = -a} = V_0 \left(1 + \frac{a}{(-a)}\right) = 0 \]

Thus, \[ \rho_1 = \rho_0 + \frac{1}{2} \rho V_0^2 \] as expected.
3.10 An incompressible irrotational steady flow past a circular cylinder as shown in Fig. P3.10. The fluid velocity along the dividing streamline \((-\infty < x < -a)\) is found to be \(V = V_0 (1 - \alpha^2 / x^2)\), where \(a\) is the radius of the cylinder and \(V_0\) is the upstream velocity. (a) Determine the pressure gradient along this streamline. (b) If the upstream pressure is \(p_0\), integrate the pressure gradient to obtain the pressure \(p(x)\) for \(-\infty < x < -a\). (c) Show from the result of part (b) that the pressure at the stagnation point \((x = -a)\) is \(p_a = \rho V_0^2 / 2\), as expected from the Bernoulli equation.

\[
\frac{df}{dx} = -\alpha^2 \sin \theta - \rho V_0 \frac{dV}{dx} \quad \text{but } \theta = 0 \quad \text{and} \quad \frac{dV}{dx} = \frac{\partial V_0}{\partial x} = \frac{2V_0}{x}
\]

Thus,

\[
\frac{df}{dx} = -\rho V_0 \frac{dV_0}{dx} = -2 \rho \alpha^2 V_0^2 \left[1 - \left(\frac{\alpha}{x}\right)^2\right] / x^3
\]

(b) \(\int_{p_0}^{p} dp = \int_{x=-\infty}^{x} \frac{df}{dx} dx\) or \(p - p_0 = -2 \rho \alpha^2 V_0^2 \left[1 - \left(\frac{\alpha}{x}\right)^2\right] \frac{dx}{x^3}\)

Thus,

\[
p = p_0 + \rho V_0^2 \left[\frac{x^2}{2} - \frac{1}{x} + \frac{\alpha^2}{x^2}\right]\quad \text{for } -\infty < x < -a
\]

(c) For \(x = -a\) from part (b):

\[
p\bigg|_{x=-a} = p_0 + \rho V_0^2 \left[-1^2 - \frac{1}{-1} + \frac{\alpha^2}{-1^2}\right] = p_a + \frac{1}{2} \rho V_0^2
\]

Note: Bernoulli equation from point (a) where \(V_1 = V_0, \ p_1 = p_0\) and \(z_1 = z_0\) to point (b) where \(V_2 = 0\), \(z_2 = z_0\) gives

\[
p_1 + \frac{1}{2} \rho V_1^2 + \delta z_1 = p_2 + \frac{1}{2} \rho V_2^2 + \delta z_2
\]

Thus,

\[
p_a = p_0 + \frac{1}{2} \rho V_0^2
\]
3.33 Water flows from the faucet on the first floor of the building shown in Fig. P3.33 with a maximum velocity of 20 ft/s. For steady inviscid flow, determine the maximum water velocity from the basement floor and from the faucet on the second floor (assume each floor is 12 ft tall).

\[ \frac{P_0}{\gamma} + \frac{V^2}{2g} + Z = \text{constant} \]

Thus, \[ \frac{P_1}{\gamma} + \frac{V_1^2}{2g} + Z_1 = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + Z_2 \]
with \( P_2 = P_1 = 0 \) (free jet) and \( V_1 = 20 \text{ ft/s} \), \( Z_1 = 4 \text{ ft} \)

or \[ \frac{(20 \text{ ft/s})^2}{2(32.2 \text{ ft/s}^2)} + 4 \text{ ft} = \frac{V_2^2}{2(32.2 \text{ ft/s}^2)} + (-8 \text{ ft}) \]

or \[ V_2 = \frac{34.2 \text{ ft/s}}{5} \]

and \[ \frac{P_1}{\gamma} + \frac{V_1^2}{2g} + Z_1 = \frac{P_3}{\gamma} + \frac{V_3^2}{2g} + Z_3 \]
with \( P_3 = P_1 = 0 \) (free jet) and \( V_1 = 20 \text{ ft/s} \), \( Z_1 = 4 \text{ ft} \)

or \[ \frac{(20 \text{ ft/s})^2}{2(32.2 \text{ ft/s}^2)} + 4 \text{ ft} = \frac{V_3^2}{2(32.2 \text{ ft/s}^2)} + 16 \text{ ft} \]

or \[ V_3 = \sqrt{20^2 - 2(32.2)(12)} = \sqrt{-373} \]

Impossible! No flow from second floor faucet.
3.51 Water flows through the pipe contraction shown in Fig. P3.51. For the given 0.2-m difference in manometer level, determine the flowrate as a function of the diameter of the small pipe, \( D \).

\[
\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2, \quad \text{or with} \quad z_1 = z_2 \quad \text{and} \quad V_1 = 0
\]

\[
V_2 = \sqrt{2g \left( \frac{\rho_1 - \rho_2}{\gamma} \right)}
\]

but \( \rho_1 = \delta h_1 \) and \( \rho_2 = \delta h_2 \) so that \( \rho_1 - \rho_2 = \delta(h_1 - h_2) = 0.2 \delta \)

Thus,

\[
V_2 = \sqrt{2g \cdot \frac{0.2 \delta^3}{\gamma}} = \sqrt{2g \cdot (0.2)}
\]

or

\[
Q = A_2 V_2 = \frac{\pi}{4} D^2 V_2 = \frac{\pi}{4} D^2 \sqrt{2 \cdot (0.2)} = 1.56 D^2 \cdot \frac{\delta^3}{\gamma} \quad \text{when} \quad D \sim m
\]
3.52 Water flows through the pipe contraction shown in Fig. P3.52. For the given 0.2-m difference in the manometer level, determine the flowrate as a function of the diameter of the small pipe, \( D \).

\[
\frac{\rho_i}{\gamma} + \frac{V_i^2}{2g} + z_i = \frac{\rho_2}{\gamma} + \frac{V_2^2}{2g} + z_2 \quad \text{with} \quad A_iV_i = A_2V_2
\]

Thus, with \( z_1 = z_2 \),

\[
\frac{\rho_1 - \rho_2}{\gamma} = \frac{V_2^2 - V_i^2}{2g} = \frac{[(\text{in}^{-1})^4 - 1]V_i^2}{2g}
\]

but \( \rho_1 = \gamma h_1 \) and \( \rho_2 = \gamma h_2 \) so that \( \rho_1 - \rho_2 = \gamma (h_1 - h_2) = 0.2 \gamma \)

Thus,

\[
0.2 \gamma = \frac{[(\text{in}^{-1})^4 - 1]V_i^2}{2g} \quad \text{or} \quad V_i = \sqrt{\frac{0.2 (2g)}{[(\text{in}^{-1})^4 - 1]}}
\]

and

\[
Q = A_iV_i = \frac{\pi}{4} (0.1)^2 \sqrt{\frac{0.2 (2 g (0.81))}{[(\text{in}^{-1})^4 - 1]}}
\]

or

\[
Q = \frac{0.015 \pi D^2}{\sqrt{(0.01)^4 - D^4}} \quad \text{m}^3/\text{s} \quad \text{when} \ D \sim \text{m}
\]
3.53 Water flows through the pipe contraction shown in Fig. P3.53. For the given 0.2-m difference in the manometer level, determine the flow rate as a function of the diameter of the small pipe, \( D \).

\[
\frac{\rho_1}{\rho} + \frac{V_1^2}{2g} + Z_1 = \frac{\rho}{\rho} + \frac{V_2^2}{2g} + Z_2
\]

where \( Z_1 = Z_2 \) and \( V_2 = 0 \).

Thus,

\[
\frac{\rho_1}{\rho} + \frac{V_1^2}{2g} = \frac{\rho_2}{\rho}
\]

But

\[
\frac{\rho_1}{\rho} = \chi \quad \text{and} \quad \frac{\rho_2}{\rho} = 0.2m + \chi \quad \text{so that}
\]

\[
\chi + \frac{V_1^2}{2g} = 0.2m + \chi \quad \text{or}
\]

\[
V_1 = \sqrt{2g(0.2m)} = \sqrt{2(981 \text{ m/s}^2)(0.2m)} = 1.98 \text{ m/s}
\]

Thus,

\[
Q = A_1 V_1 = \frac{\pi}{4} (0.1m)^2 (1.98 \text{ m/s}) = 0.0156 \text{ m}^3/\text{s}
\]

for any \( D \).
If viscous effects are neglected and the tank is large, determine the flow rate from the tank shown in Fig. P3.72

\[ \frac{\rho_1}{g} + \frac{V_1}{2g} + Z_1 = \frac{\rho_2}{g} + \frac{V_2}{2g} + Z_2 \]

where \( \rho_1 = \rho_0 + \Delta \rho = \rho h \)

\( Z_1 = 0.7 \text{ m}, \ Z_2 = 0, \text{ and } V_1 = 0 \)

Thus,

\[ \frac{\rho h}{g} + Z_1 = \frac{V_2}{2g} \quad \text{or} \quad V_2 = \sqrt{2g \left( \frac{\rho h}{g} + Z_1 \right)} \]

where \( \frac{\rho h}{g} = 0.81 \)

and

\[ Q = A_2 V_2 = \frac{\pi}{4} a^2 V_2 \]

Thus,

\[ Q = \frac{\pi}{4} (0.05 \text{ m})^2 \sqrt{2(9.81 \text{ m/s}^2)(0.81(2 \text{ m}) + 0.7 \text{ m})} = 0.0132 \text{ m}^3/\text{s} \]
Air flows readily through the variable area pipe shown in Fig. P3.81. Determine the flowrate if viscous and compressibility effects are negligible.

From the Bernoulli equation,

(1) \[ \frac{\rho_1}{\rho_{air}} + \frac{V_1^2}{2g} + Z_1 = \frac{\rho_2}{\rho_{air}} + \frac{V_2^2}{2g} + Z_2 \]

where \( Z_1 = Z_2 \) and \( V_2 = 0 \)

and

(2) \[ Q = A_1 \sqrt{\frac{\rho_1}{\rho_{air}}} \]

Also, from the manometer

(3) \[ \rho_1 + \delta_{h,20} h_1 + \delta_{h,20} h_1 = \rho_2 + \delta_{air} (h + h_1) \]

But \( \delta_{h,20} \gg \delta_{air} \) so that Eq. (3) becomes

\[ \rho_2 = \rho_1 + \delta_{h,20} h_1 \quad \text{or} \quad V_1^2 = \frac{\rho_1}{\rho_{air}} + \frac{\delta_{h,20}}{\delta_{air}} h_1 \]

Hence, from Eq. (1):

\[ \frac{\rho_1}{\rho_{air}} + \frac{V_1^2}{2g} = \frac{\rho_1}{\rho_{air}} + \left( \frac{\delta_{h,20}}{\delta_{air}} \right) h_1 \]

or

\[ V_1 = \sqrt{2g \left( \frac{\delta_{h,20}}{\delta_{air}} \right) h_1} = \sqrt{2 \left( 9.8 \text{ m/s}^2 \right) \left( \frac{9.8 \times 10^3 \text{ N/m}^2}{12.0 \text{ N/m}^3} \right) (0.1 \text{ m})} = 40.0 \text{ m/s} \]

Thus, from Eq. (2),

\[ Q = \frac{\pi}{4} (0.2 \text{ m})^2 (40.0 \text{ m/s}) = 1.2 \text{ m}^3/\text{s} \]