MAE 3360 ENGINEERING ANALYSIS II
SUMMER 2009
DEPARTMENT OF MECHANICAL
AND
AEROSPACE ENGINEERING
UNIVERSITY OF TEXAS AT ARLINGTON

Exam #2

July 29, 2009
9:00-10:45 am

CLOSED BOOKS
Non-programmable calculators are allowed

This exam has a total of 7 pages.

Show all your work
Illegible writings or incomplete explanations
will result in loss of points

LAST NAME: [Signature]

FIRST NAME: ____________________
1. (20 pts) Find the general solution of: \( \frac{d^2y}{dx^2} + y = \sec x \)

\[ y_h : e^{mx} \implies m^2 + 1 = 0 \implies m = \pm i \]

\[ y_h = (c_1 \cos x + c_2 \sin x) \]

\[ y_p : (\text{variation of parameters}) \]

\[ W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1 \]

\[ y_p = u_1 y_1 + u_2 y_2 \]

\[ u'_1 = \begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix} = -\sin x \sec x = -\frac{\sin x}{\cos x} \]

\[ u'_1 = \int -\frac{\sin x}{\cos x} \, dx = \int \frac{\cos x}{\cos x} \, dx = \ln (\cos x) \]

\[ u'_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \sin x \end{vmatrix} = 1 \implies u_2 = x \]

\[ y_p = u_1 y_1 + u_2 y_2 = \cos x \ln (\cos x) + x \sin x \]

\[ y = y_h + y_p = (c_1 \cos x + c_2 \sin x) + \cos x \ln (\cos x) + x \sin x \]
2.(20 pts) Find the general solution of: \( x^2 \frac{d^2 y}{dx^2} - 4x \frac{dy}{dx} + 6y = \frac{1}{x} \)

\[
y'' - \frac{4}{x} y' + \frac{6}{x^2} y = \frac{1}{x^3}
\]

Let \( y_m = x^m \) \(\Rightarrow\) \( m(m-1) - 4m + 6 = 0 \)
\(\Rightarrow\) \( m^2 - 5m + 6 = 0 \)
\(\Rightarrow\) \( m = 2, 3 \)

\[
y_m = c_1 x^2 + c_2 x^3
\]

\( Y_1 = y_1, \quad Y_2 = y_2 \)

\[
W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = x^4
\]

\[
u' = \frac{1}{x^4} \begin{vmatrix} 0 & x^3 \\ x & 3x^2 \end{vmatrix} = \frac{1}{x^5} \quad \Rightarrow \quad u = \frac{1}{4} x^{-3}
\]

\[
u' = \frac{1}{x^4} \begin{vmatrix} x^2 & 0 \\ 2x & x^3 \end{vmatrix} = \frac{1}{x^5} \quad \Rightarrow \quad \nu = \frac{1}{4} x^{-4}
\]

\[
y = y_m + y_p = u y_1 + v y_2 = \frac{1}{3} x^{-3} + \frac{1}{4} x^{-4}
\]

\[
\int y = y_h + y_p = c_1 x^2 + c_2 x^3 + \frac{1}{12x}
\]
3. (20pts) Consider the damped spring-mass system whose motion is governed by
\[
\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 10\sin t; \quad y(0) = 1, \quad y'(0) = 0
\]
(a) Determine whether the motion is underdamped, overdamped or critically damped.
(b) Determine the amplitude of the steady-state oscillation.

(a) 
\[
m^2 + 3m + 2 = 0 \quad \Rightarrow \quad m = -1, -2
\]
\[
y_h = C_1e^{-t} + C_2e^{-2t}
\]

Clearly the system is overdamped, i.e. no sinusoidal oscillations!

(b) 
\[
y_p = A\sin t + B\cos t
\]
\[
y_p' = A\cos t - B\sin t
\]
\[
y_p'' = -A\sin t - B\cos t
\]

Sub. y'' = 10\sin t 

\[
\begin{cases}
-A - 3B + 2A = 10 \\
-B + 3A + 2B = 0
\end{cases}
\]

\[
\Rightarrow \begin{cases}
A = 1 \\
B = -3
\end{cases}
\]

\[
y_p = \sin t - 3\cos t
\]

\[
\Rightarrow \quad \text{Amplitude of steady-state oscillation} = \sqrt{1^2 + 3^2} = \sqrt{10}
\]

Note: I.C.'s are not needed as the \( y_h \) is not needed for the steady state solution.
A mass weighing 32 pounds is attached to a spring whose spring constant is 5 lb/ft. Initially the mass is released from rest from a point 1 foot below the equilibrium position, and the subsequent motion takes place in a medium that offers a damping force numerically equal to twice the instantaneous velocity. Determine the equation of motion if the mass is driven by an external force, in pounds, equal to \(10 \cos 3t\). Explain clearly the formulation of the problem.

\[
W = 32 \text{ lb} \Rightarrow m = \frac{32}{g}
\]

\[
k = 5 \text{ lb/ft}
\]

\[
\beta = 2
\]

\[
m \dddot{x} + \beta \dot{x} + kx = 10 \cos 3t \\
(\text{in pounds})
\]

\[
\dddot{x} + 2 \ddot{x} + 5x = 10 \cos 3t \quad (1)
\]

\[
\begin{align*}
\text{I.C.} \quad x(0) &= 1 \\
\dot{x}(0) &= 0
\end{align*}
\]

\[
x_h = \dddot{x} + 2 \ddot{x} + 5x = 0 \Rightarrow m^2 + 2m + 5 = 0
\]

\[
m = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 5}}{2}
\]

\[
x_h = e^{-t} \left( C_1 \cos 2t + C_2 \sin 2t \right)
\]

\[
x_p = \dot{x}_p = A \cos 3t + B \sin 3t
\]

\[
\begin{align*}
\dot{x}_p &= -3A \sin 3t + 3B \cos 3t \\
\ddot{x}_p &= -9A \cos 3t - 9B \sin 3t
\end{align*}
\]

\[
\begin{align*}
6B - 4A &= 10 \\
-4B - 6A &= 0
\end{align*}
\]

\[
\Rightarrow A = \frac{-10}{13} \quad B = \frac{15}{13}
\]

\[
x = x_h + x_p = e^{-t} \left( C_1 \cos 2t + C_2 \sin 2t \right) + \frac{15}{13} \sin 3t - \frac{10}{13} \cos 3t
\]

\[
x(0) = 1 \Rightarrow C_1 = 1 - \frac{5 \cdot 10}{13} \Rightarrow C_1 = \frac{10}{13}
\]

\[
\dot{x}(0) = 0 \Rightarrow \left[ C_2 = -\frac{11}{13} \right]
\]
5a. (10pts) Using the fourth-order Runge-Kutta formula, with \( \Delta t = 0.1 \), determine the solution of \( y \), up to the fourth decimal place, at \( t = 0.1 \) for the following initial value problem:

\[
\frac{dy}{dt} = 1 + y^2; \quad y(0) = 0
\]

Show clearly details of all computations.

\[
f(t, y) = 1 + y^2
\]

\( t_0 = 0 \), \( y_0 = 0 \)

\( h = \Delta t = 0.1 \)

\[
k_1 = f(t_0, y_0) = 1 + y_0^2 = 1
\]

\[
k_2 = f(t_0 + \frac{\Delta t}{2}, y_0 + \frac{k_1 h}{2}) = f(0.05, 0.05)
\]

\[
k_3 = f(t_0 + \frac{\Delta t}{2}, y_0 + \frac{k_2 h}{2}) = 1 + 0.05^2 = 1.0025
\]

\[
k_4 = f(t_0 + \Delta t, y_0 + k_3 h) = 1.0100
\]

\[
y_1 = y_0 + \frac{h}{6} (k_1 + 2k_2 + 2k_3 + k_4)
\]

\[
y_1 = 0.1003
\]
5b. (10pts) Using the Euler's method, with $\Delta t = 0.1$, determine the solution of $y$, up to the fourth decimal place, at $t = 0.2$ for the following initial value problem:

$$\frac{d^2y}{dt^2} + t \frac{dy}{dt} + 2y = 0 ; \quad y(0) = 1, \quad \frac{dy}{dt}(0) = 3$$

Show details of all computations and enter the values of $y$ in the table below.

<table>
<thead>
<tr>
<th>$t$</th>
<th>0.1</th>
<th>0.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>1.3</td>
<td>1.58</td>
</tr>
</tbody>
</table>

\[
y' = y + 2y = 0 \quad y' = u \Rightarrow u' = -tu - 2y
\]

$t_0 = 0$, $y_0 = 1$; $y_0' = 3 \Rightarrow u_0 = 3$

$\Delta t = 0.1$

\[
y_1 = y_0 + \Delta t \cdot u_0 = 1 + (0.1)(3) = \boxed{1.3}
\]

\[
u_1 = u_0 + \Delta t \left(-t_0 \cdot u_0 - 2y_0\right)
= 3 + 0.1 \left(-0 \cdot 3 - 2 \cdot 1\right) = 2.8
\]

\[
y_2 = y_1 + \Delta t \cdot u_1 = 1.3 + (0.1)(2.8) = \boxed{1.58}
\]