Introduction to Ordinary Differential Equation:

First Order & Homogeneous ODEs:

Exact Differential Methods:

For given ODE as

\[ F(x,y) \, dx + G(x,y) \, dy = 0 \]

The condition for exact differential is

\[ \frac{\partial F}{\partial y} = \frac{\partial G}{\partial x} \]

Example: \((x + \sin y) \, dx + (x \cos y - 2y) \, dy = 0\)

Now, \( F(x,y) = x + \sin y \)

\( G(x,y) = x \cos y - 2y \)

\[ \frac{\partial F}{\partial y} = \cos y = \frac{\partial G}{\partial x} \]

The given ODE is exact differential equation.

Now, \( U(x,y) = \int (x + \sin y) \, dx + G(x) \)

\[ = \frac{x^2}{2} + x \sin y + \int G(x) \]

\[ U(x,y) = \int (x \cos y - 2y) \, dy + g(x) \]

\[ = x \sin y - y^2 + g(x) \]
\[ y = \pm \sqrt{e^x(x-1)+C} \] + solution

**Beroulli's Equation:**

If the ODE is of the form

\[ \frac{dy}{dx} + P(x) y = Q(x) y^n \; ; \; n \neq 1 \]

then use the substitution as

\[ v = y^{1-n} \]

and solve the ODE in terms of \( v \) and finally substitute \( v = y^{2-n} \) to get answer in terms of \( y \).

**Homogeneous Equations/Substitution Methods:**

If the given ODE is Homogeneous to \( n^{th} \) degree \( (n = \text{const}) \), then use the substitution \( \frac{x}{y} = v \) for \( y = vx \) and solve the equation.

→ For example, refer to the SI notes on 64th Feb 2008.
\[ 
\frac{1}{x} \frac{dy}{dx} - \frac{y}{x^2} = 0 
\]

\[ 
\therefore \frac{d}{dx} \left( \frac{y}{x} \right) = 0 
\]

Now integrating both sides,

\[ 
y/x = c 
\]

\[ 
\Rightarrow \left[ y = cx \right] 
\]

* Separation of Variables Method:*

If the given ODE is of

\[ 
f(x, y) \, dx + g(x, y) \, dy = 0 
\]

\[ 
f(x, y) = F_1(x), F_2(y) 
\]

\[ 
g(x, y) = G_1(x), G_2(y) 
\]

Then,

\[ 
\frac{F_1(x)}{G_1(x)} \, dx + \frac{G_2(y)}{F_2(y)} \, dy = 0
\]

Now integrating both sides we can get the solution to be

\[ 
y = xe^x 
\]

\[ 
\Rightarrow \quad \frac{dy}{dx} = \frac{xe^x}{y} 
\]

Using separation of variables method,

\[ 
y \, dy = xe^x \, dx 
\]

Now integrating both sides,

\[ 
\int 2y \, dy = \int xe^x \, dx + c 
\]

\[ 
\therefore \quad y^2 = xe^x - e^x + c 
\]
Now, comparing (i) & (iii) we have,

\[ f(y) = -y^2 \quad \text{and} \quad g(x) = x^{\frac{3}{2}} \]

1. \[ U(x, y) = \frac{x^3}{6} + x \sin y - y^2 \]

Now, solution is \[ U(x, y) = \frac{x^3}{6} + x \sin y - y^2 = C \]

**Integrating Factor Method:**

If the given ODE is not exact and of the form

\[ \frac{dy}{dx} + g(x)y = f(x) \]

Find an integrating factor \( I.F. = \rho(x) = e^{\int g(x) \, dx} \)

Now multiply with integrating factor to the given equation.

\[ \rho(x) \frac{dy}{dx} + \rho(x) g(x)y = \rho(x)f(x) \]

**Example:** \[ y \, dx - x \, dy = 0 \]

\[ \Rightarrow \quad \frac{dy}{dx} + \frac{g(x)}{f(x)} = 0 \]

\[ \Rightarrow \quad \text{It is a non-exact differential.} \]

So, Now, \[ -x \frac{dy}{dx} + y = 0 \]

\[ \Rightarrow \quad \frac{dy}{dx} - \frac{y}{x} = 0 \]

Now, \[ \rho(x) = e^{\int -\frac{1}{x} \, dx} = e^{-\ln x} = \frac{1}{x} = \frac{1}{x} \]
Solution of linear ODEs with constant coefficients:

General Approach:

1. First obtain the characteristic equation from the given ODE with constant coefficients.
2. Solve the quadratic characteristic equation.  
   3. Possibilities:

   (I) Two distinct real roots
   \[ y = A_1 e^{m_1t} + A_2 e^{m_2t} \] 
   \[ m_1, m_2 \text{ - roots} \]
   \[ A_1, A_2 \text{ - constant} \]

   (II) Repeated roots
   \[ y = A_1 e^{mt} + B A_2 e^{mt} \] 
   \[ m \text{ - repeated roots} \]

   (III) Two complex conjugate roots
   \[ y = e^{at} (A \cos bt + A \sin bt) \] 
   \[ m = a + ib \]

Example: \[ y + 10y + 21y = 0 \]

Solution: The given ODE is \[ y + 10y + 21y = 0 \]

The characteristic equation from the given ODE is,
\[ m^2 + 10m + 21 = 0 \]
\[ (m+7)(m+3) = 0 \]
\[ m = -7, -3 \text{ two distinct real roots} \]
Example 2: \( y'' + 4y' + 5y = 0 \)

The characteristic equation from the given ODE is:

\[ m^2 + 4m + 5 = 0 \]

\[ \Rightarrow m = \frac{-4 \pm \sqrt{16 - 20}}{2} \]

\[ = -2 \pm i \]

So, \( m = -2 \pm i \) are complex conjugate roots.

Solution:

\[ y = Ae^{-2t} (A_1 \cos t + A_2 \sin t) \]
Equidimensional Equations: (Linear ODE with Variable Co-eff.)

If the given ODE is of the general form like,

\[ a^n \frac{d^n y}{dx^n} + b_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + b_1 \frac{dy}{dx} + b_0 y = 0 \]

all b's are constant.

→ To solve such equation, take \( y = x^m \).

and then find out the \( \frac{d y}{d x} \), \( \frac{d^2 y}{d x^2} \), \ldots \( \frac{d^m y}{d x^m} \) and

plug-in those values to the given ODE.

→ Find out the characteristic equation for the above
domain and solve it being quadratic equation
solution.

→ Substitute values of roots to the trial solution
\( x^n \) and get the final solution.

→ For examples of such ODE refer to the 51 notes
06 13th February, 2008.
Solution of Non-Homogeneous Equations:

To solve a Non-Homogeneous ODE equation, solution:

\[ y'' + y' + y = f(x) \]

1. Find out the solution of Homogeneous Equation, \( y_h \) by making \( L[y] = 0 \).

2. Find out the Particular Solution, \( y_p \) by using the Method of Undetermined Coefficients or for the cases where a family can be found easily for non-homogeneous terms.

3. General solution will be the sum of the Homogeneous & Particular solution.

\[ y = y_h + y_p \]

Method of Undetermined Coefficients for \( y_p \):

- First decide the family (check table in textbook for some families).
- Take trial \( y_p \) from the chosen family.
- Plug it in the given ODE and by using comparison find out the coefficients (unknown) of \( y_p \).

Note: If you find the exact same term in the \( y_p \) as it is in \( y_h \) then modify the \( y_p \) until it doesn't contain the exact same term of \( y_h \).
Example: \( y' - 5y = e^x - xe^{5x} \)

1. Find out \( y_h \):

\[
y' - 5y = 0
\]

\( \Rightarrow \) characteristic equation,

\[
m - 5 = 0 \quad \Rightarrow \quad m = 5
\]

\[
y_h = Ae^{5x}
\]

2. Find \( y_p \):

From non-homogeneous terms,

Family for \( x^2 e^x \)

\[
y_p_1 = e^x (Ax^2 + Bx + C)
\]

For \( x e^{5x} \) part,

\[
y_p_2 = e^{5x} (Dx + E)
\]

Now, the \( y_p_2 \) contains the term of \( y_h \) \( \Rightarrow e^{5x} \)

So we need to modify it,

\[
y_p_2 = e^{5x} (Dx^2 + Ex)
\]

Now the force term the terms of \( y_h \).

So, \( y_p = y_p_1 + y_p_2 \)

\[
y_p = e^x (Ax^2 + Bx + C) + e^{5x} (Dx^2 + Ex)
\]

Now, \( y_p' = e^x (2Ax + B) + e^x (Ax^2 + Bx + C) + e^{5x} (2Dx + E) + 5e^{5x} (Dx^2 + Ex) \)
Now from given ODE

\[ y' - 5y = x^2 e^x - x e^{5x} \]

\[ e^x (2Ax + B) + e^x (Ax^2 + Bx + C) + e^{5x} (2Dx + E) \]

\[ + 5e^{5x} ( Dx^2 + Ex ) - 5 e^x ( Ax^2 + Bx + C ) - 5 e^{5x} ( Dx^2 + Ex ) \]

\[ = x^2 e^x - x e^{5x} \]

\[ 2Ax e^x + B e^x + 4e^x ( Ax^2 + Bx + C ) + 2Dxe^{5x} + E e^{5x} \]

\[ = x^2 e^x - x e^{5x} \]

\[ 2Ax e^x + B e^x - 4Ax e^x - 4Bxe^x - 4C e^x + 2Dxe^{5x} \]

\[ + E e^{5x} - x^2 e^x - x e^{5x} \]

\[ (2A - 4B) e^x - 4Ax e^x + (B - 4C) e^x + 2Dxe^{5x} + E e^{5x} \]

\[ = x^2 e^x - x e^{5x} \]

Now, comparing both sides we have:

\[ 2A - 4B = 0 \quad B - 4C = 0 \quad E = 0 \]

\[ -4A = 1 \quad 2D = -1 \]

\[ \Rightarrow A = -\frac{1}{4} \quad D = -\frac{1}{2} \]

\[ \text{Now, } a((-\frac{1}{4} ) - 4B = 0 \]

\[ -4B = \frac{1}{2} \]

\[ \Rightarrow B = -\frac{1}{8} \]

\[ \text{Now, } -\frac{1}{8} - 4C = 0 \]

\[ -4C = \frac{1}{8} \]

\[ \Rightarrow C = -\frac{1}{32} \]
So, now,

\[ y_p = e^x \left( -\frac{1}{4} x^2 - \frac{1}{8} x - \frac{1}{32} \right) + e^{5x} \left( -\frac{1}{2} x^2 \right) \quad \text{Eq. (1)} \]

\[ y_p = e^x \left( -\frac{1}{4} x^2 - \frac{1}{8} x - \frac{1}{32} \right) - \frac{1}{2} x^2 e^{5x} \]

Now, general solution,

\[ y = y_h + y_p \]

\[ y = A_1 e^{5x} + e^x \left( -\frac{1}{4} x^2 - \frac{1}{8} x - \frac{1}{32} \right) - \frac{1}{2} x^2 e^{5x} \]

Solution.