Ex 31: A mass of 1 slug, when attached to a spring, stretches it 2 feet and then comes to rest in the equilibrium position. Starting at \( t = 0 \), an external force equal to \( f(t) = 8 \sin 4t \) is applied to the system. Find the equation of motion if the surrounding medium offers a damping force numerically equal to 8 times the instantaneous velocity.

**Solution: Given Data:**

\[ m = 1 \text{ Slug} \quad x(0) = 0 \]
\[ k = 2 \text{ ft} \quad x'(0) = 0 \]
\[ f(t) = 8 \sin 4t \quad \text{Damping} = 8 x' \quad \implies B = 8 \]

Now, we know that \( mg = k \delta \)

\[ k = \frac{mg}{\delta} = \frac{1 \times 32}{2} = 16 \text{ lb/ft} \]

\[ \omega^2 = \frac{k}{m} = \frac{16}{1} = 16 \]

\[ 2\lambda = \frac{B}{m} = \frac{8}{1} = 8 \]

Now, the system is described by the equation,

\[
\begin{align*}
\frac{d^2x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x &= f(t) \\
\frac{d^2x}{dt^2} + 8 \frac{dx}{dt} + 16x &= 8 \sin 4t
\end{align*}
\]
Now, solution of eqn (x) is as follows....

**Homogeneous solution (xₙ):**

\[ \frac{d^2x}{dt^2} + 8 \frac{dx}{dt} + 16x = 0 \]

*Characteristic equation,*

\[ M^2 + 8M + 16 = 0 \]

\[ \Rightarrow (M + 4)^2 = 0 \]

\[ \Rightarrow M = -4 \] a double root

\[ \therefore Xₙ(t) = c₁e^{-4t} + c₂te^{-4t} \]

**Particular solution (xₚ):**

Let, \[ Xₚ(t) = A \sin 4t + B \cos 4t \]

\[ Xₚ'(t) = 4A \cos 4t - 4B \sin 4t \]

\[ Xₚ''(t) = -16A \sin 4t - 16B \cos 4t \]

So, now, \[ Xₚ'' + 8Xₚ' + 16Xₚ = 8 \sin 4t \]

\[ \Rightarrow -16A \sin 4t - 16B \cos 4t + 32A \cos 4t - 32B \sin 4t \]

\[ + 416A \sin 4t + 16B \sin 4t = 8 \sin 4t \]

Comparing both sides,

\[ -32B = 8 \quad \Rightarrow \quad B = -\frac{1}{4} \]

\[ 82A = 0 \quad \Rightarrow \quad A = 0 \]

\[ \therefore Xₚ(t) = -\frac{1}{4} \cos 4t \]

**Now general solution,**

\[ X(t) = Xₙ(t) + Xₚ(t) \]

\[ X(t) = c₁e^{-4t} + c₂te^{-4t} - \frac{1}{4} \cos 4t \]
Now, applying Initial conditions...

\[ x(0) = 0 = c_1 + \frac{-1}{4} \implies c_1 = \frac{1}{4} \]
\[ x'(t) = -4c_1 e^{-4t} - 4c_2 t e^{-4t} + c_2 e^{-4t} + \sin 4t \]
\[ x'(0) = 0 = -4c_1 + c_2 = -1 + c_2 \]
\[ \implies c_2 = 1 \]

so, \[ x(t) = \frac{1}{4} e^{-4t} + te^{-4t} - \frac{1}{4} \cos 4t \] is the solution.

\[ \text{UNDAMPED FORCED MOTION} \]

In the undamped forced motion system there will be no damping force, so the equation of motion becomes (i.e.):

\[ \frac{d^2x}{dt^2} + \omega^2 x = f(t) \]

\[ \text{Ex: Show that the solution of the initial-value problem} \]
\[ \frac{d^2x}{dt^2} + \omega^2 x = F_0 \cos \omega t, \quad x(0) = 0, \quad x'(0) = 0 \]
\[ \text{is} \quad x(t) = \frac{F_0}{\omega^2 - \gamma^2} \left( \cos \omega t - \cos \omega t \right) \]

(b) Evaluate \( \lim_{\gamma \to 0} \frac{F_0}{\omega^2 - \gamma^2} \left( \cos \omega t - \cos \omega t \right) \)
Solution (a)

\[ \frac{d^2 x}{dt^2} + \omega^2 x = F_0 \cos \gamma t \]

It is the 2nd order ODE. 

Homogeneous solution \((x_h)\)

\[ \frac{d^2 x}{dt^2} + \omega^2 x = 0 \]

Characteristic Equation,

\[ M^2 + \omega^2 = 0 \]

\[ \Rightarrow M = \pm \omega i \]

\[ x_h(t) = C_1 \cos \omega t + C_2 \sin \omega t \]

Particular solution \((x_p)\)

Let,

\[ x_p(t) = A \cos \gamma t + B \sin \gamma t \]

\[ x_p' = -\gamma A \sin \gamma t + \gamma B \cos \gamma t \]

\[ x_p'' = -\gamma^2 A \cos \gamma t - \gamma^2 B \sin \gamma t \]

Now,

\[ x_p'' + \omega^2 x_p = F_0 \cos \gamma t \]

\[ -\gamma^2 A \cos \gamma t - \gamma^2 B \sin \gamma t + \omega^2 A \cos \gamma t + \omega^2 B \sin \gamma t \]

\[ = F_0 \cos \gamma t \]

\[ \Rightarrow A(\omega^2 - \gamma^2) \cos \gamma t + B(\omega^2 - \gamma^2) \sin \gamma t = F_0 \cos \gamma t \]

both side comparing \(\omega^2 - \gamma^2\) co-eff.

\[ A(\omega^2 - \gamma^2) = F_0 \]

\[ \Rightarrow A = \frac{F_0}{\omega^2 - \gamma^2} \]

\[ B(\omega^2 - \gamma^2) = 0 \]

\[ \Rightarrow B = 0. \]
\[ x(t) = x_n(t) + x_p(t) \]

\[ x(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{F_0}{\omega^2 - \gamma^2} \cos \gamma t \]

Now applying initial conditions,

\[ x(0) = 0 = c_1 + \frac{F_0}{\omega^2 - \gamma^2} \Rightarrow c_1 = \frac{F_0}{\gamma^2 - \omega^2} \]

\[ x'(0) = 0 = \omega c_1 \sin \omega t + \omega c_2 \cos \omega t = \frac{F_0 \gamma}{\omega^2 - \gamma^2} \sin \gamma t \]

\[ x'(0) = 0 = \omega c_1 \sin \omega t + \omega c_2 \cos \omega t = 0 \Rightarrow c_2 = 0 \]

\[ x(t) = \frac{F_0}{\gamma^2 - \omega^2} \left( \cos \omega t - \cos \gamma t \right) \]

(b)

\[ \lim_{\gamma \to \omega} \frac{F_0}{\omega^2 - \gamma^2} \left( \cos \omega t - \cos \omega t \right) \]

Applying L'Hopital's rule,

\[ \lim_{\gamma \to \omega} -\frac{F_0 \gamma \sin \gamma t}{-2\gamma} \]

\[ = \frac{F_0}{2\omega} \sin \omega t \]
SI Session
MAE 3260

SI Leaders: Monalikumar Patel
Date: 03/24/2008

* Numerical Methods to solve ODE's:

* Taylor's Series: Taylor's series is defined as

\[
f(x+h) = f(x) + h \cdot f'(x) + \frac{h^2}{2!} f''(x) + \ldots +
\]

\[
\ldots + \frac{h^n}{n!} f^{(n)}(x) + \ldots
\]

* Euler's Method (first order):

To solve the ODE of 1st order

\[
y' = f(x, y) ; \quad y(x_0) = y_0
\]

we can use this method.

- For a given step size 'h' we can find...

\[
x_1 = x_0 + h,
\]
\[
x_2 = x_1 + h = x_0 + 2h,
\]
\[
\vdots
\]
\[
x_n = x_0 + nh.
\]

Now, from Taylor's series

\[
y(x+h) \approx y(x) + h \cdot y'(x)
\]

\[
\text{[Neglecting higher order terms]}
\]

\[
y(x+h) = y(x) + h \cdot f(x, y)
\]
\[ y_{n+1} = y_n + h \cdot f(x_n, y_n) \]

Find \( y(0.5) \) for \( y' = 2x \); \( y(0) = 0 \) using Euler's method with \( h = 0.1 \).

**Solution:**

**Given:**

\[ y' = 2x, \quad y(0) = 0, \quad h = 0.1 \]

\[ f(x, y) = 2x \]

\[ \begin{align*}
  h &= 0 \\
  \{ x_0 &= 0 \\
  y_0 &= 0 \\
  f(x_0, y_0) &= 2(0) = 0.
\end{align*} \]

\[ h = 0.1 \]

\[ \begin{align*}
  x_1 &= x_0 + h = 0.1 \\
  y_1 &= y_0 + h \cdot f(x_0, y_0) \\
  y_1 &= 0 + 0.1(0) = 0 \\
  f(x_1, y_1) &= 2x_1 = 2(0.1) = 0.2
\end{align*} \]

\[ h = 0.2 \]

\[ \begin{align*}
  x_2 &= x_1 + h = 0.1 + 0.1 = 0.2 \\
  y_2 &= y_1 + h \cdot f(x_1, y_1) \\
  y_2 &= 0 + 0.1(0.2) = 0 + 0.02 \\
  y_2 &= 0.02 \\
  f(x_2, y_2) &= 2x_2 = 2(0.2) = 0.4
\end{align*} \]
\[ n = 3: \]
\[ x_3 = x_2 + h = 0.3 \]
\[ y_3 = y_2 + h \cdot f(x_2, y_2) \]
\[ = 0.02 + 0.1(0.4) \]
\[ y_3 = 0.06. \]
\[ f(x_3, y_3) = 2x_3 = 2(0.3) = 0.6. \]

\[ n = 4: \]
\[ x_4 = x_3 + h = 0.4 \]
\[ y_4 = y_3 + h \cdot f(x_3, y_3) \]
\[ = 0.06 + 0.1(0.6) \]
\[ y_4 = 0.12 \]
\[ f(x_4, y_4) = 2x_4 = 0.8. \]

\[ n = 5: \]
\[ x_5 = x_4 + h = 0.5 \]
\[ y_5 = y_4 + h \cdot f(x_4, y_4) \]
\[ = 0.12 + 0.1(0.8) \]
\[ y_5 = 0.20 \]

Now, we can form the table of result as below:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_n )</th>
<th>( y_n )</th>
<th>Exact Sold. ( (y = x^2) )</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0.1</td>
<td>0</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td>2</td>
<td>0.2</td>
<td>0.02</td>
<td>0.04</td>
<td>0.02</td>
</tr>
<tr>
<td>3</td>
<td>0.3</td>
<td>0.06</td>
<td>0.09</td>
<td>0.03</td>
</tr>
<tr>
<td>4</td>
<td>0.4</td>
<td>0.12</td>
<td>0.16</td>
<td>0.04</td>
</tr>
<tr>
<td>5</td>
<td>0.5</td>
<td>0.20</td>
<td>0.25</td>
<td>0.05</td>
</tr>
</tbody>
</table>
Find $y(0.6)$ for $y' = -y + x + 2$; $y(0.3) = 2.0351$ with $h = 0.05$ and compare results with exact solution.

**Solution:**

Given $f(x, y) = -y + x + 2$.

**Step 1 ($n=0$):**

$x_0 = 0.3$

$y_0 = 2.0351$

$f(x_0, y_0) = -y_0 + x_0 + 2 = 0.2649$

**Step 2 ($n=1$):**

$x_1 = x_0 + h = 0.35$

$y_1 = y_0 + h \cdot f(x_0, y_0)$

$= 2.0351 + 0.05(0.2649)$

$= 2.048345$

$f(x_1, y_1) = -y_1 + x_1 + 2 = 0.361655$

**Step 3 ($n=2$):**

$x_2 = x_1 + h = 0.4$

$y_2 = y_1 + h \cdot f(x_1, y_1)$

$= 2.048345 + 0.05(0.361655)$

$= 2.06343$

$f(x_2, y_2) = -y_2 + x_2 + 2 = 0.33657$

**Step 4 ($n=3$):**

$x_3 = x_2 + h = 0.45$

$y_3 = y_2 + h \cdot f(x_2, y_2)$

$= 2.06343 + 0.05(0.33657)$

$= 2.08026$
\[ f(x_3, y_3) = -y_3 + x_3 + 2 = 0.36974. \]

\[ n = 4 \]

\[ x_4 = x_3 + h = 0.5 \]

\[ y_4 = y_3 + hf(x_3, y_3) \]

\[ = 0.08026 + 0.05 \times (0.36974) \]

\[ y_4 = 0.09875 \]

\[ f(x_4, y_4) = -y_4 + x_4 + 2 = 0.40175 \]

\[ n = 5 \]

\[ x_5 = x_4 + h = 0.55 \]

\[ y_5 = y_4 + hf(x_4, y_4) \]

\[ = 0.09875 + 0.05 \times (0.40175) \]

\[ y_5 = 0.11881 \]

\[ f(x_5, y_5) = -y_5 + x_5 + 2 = 0.48119 \]

\[ n = 6 \]

\[ x_6 = x_5 + h = 0.6 \]

\[ y_6 = y_5 + hf(x_5, y_5) \]

\[ = 0.11881 + 0.05 \times (0.48119) \]

\[ y_6 = 0.14087 \]

\[ y(0.6) = 0.14087 \]
Exact solution

\[ y(x) = e^{-x} + x + 1. \]

\[ y(0.6) = e^{-0.6} + 0.6 + 1 \]

\[ y(0.6) = 2.14881 \]

Note: This shows that with smaller step size, the Euler's method gives more accurate result.