For the equilibrium condition, by using Hooke's Law

\[ \sum F = 0. \]

\[ \Rightarrow \text{mg} - kx = 0. \]

\[ \left[ \text{mg} = kx \right] \]

Now for free motion, using Newton's Second Law,

\[ \text{mg} - k(x + \Delta x) = m \Delta x'' \]

\[ \Rightarrow \text{mg} - ks - k\Delta x = m \frac{d^2 \Delta x}{dt^2} \]

\[ \Rightarrow m \frac{d^2 x}{dt^2} + kx = 0 \]
\[ \frac{d^2 x}{dt^2} + \frac{k}{m} x = 0 \]

\[ \frac{d^2 x}{dt^2} + \omega_0^2 x = 0 \quad \text{where, } \omega_0 = \frac{k}{m}. \]

Which is ODE and can be solved as:

\[ x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t \]

This is known as equation of motion.

→ Period is defined as \[ T = \frac{2\pi}{\omega_0}. \]

→ Frequency is defined as \[ f = \frac{1}{T} = \frac{\omega_0}{2\pi}. \]

→ The equation of motion can be also written as in the alternative form as:

\[ x(t) = A \sin(\omega_0 t + \phi) \]

Where, \( A = \sqrt{c_1^2 + c_2^2} \quad \text{Amplitude} \)

\[ \phi = \tan^{-1} \left( \frac{c_1}{c_2} \right) \quad \text{Phase angle} \]

**Example**: A mass weighing 24 pounds, attached to the end of a spring, stretches it 4 inches. Initially, the mass is released from rest from a point 3 inches above the equilibrium position. Find the equation of motion.

**Solution**: Given data:

\[ W = 24 \text{ lb} \]
\( s = 4 \text{ inch.} = \frac{9}{12} \text{ ft} = \frac{1}{3} \text{ ft} \)

\( x(0) = -3 \text{ inch.} = -\frac{3}{12} \text{ ft} = -\frac{1}{4} \text{ ft} \)

\( x'(0) = 0 \) \{ released from rest \}

\[ \Rightarrow \text{ Now, we have } \quad mg = ks \]

\[ \Rightarrow 24 = k \left( \frac{1}{3} \right) \]

\[ \Rightarrow k = 72 \text{ lb/ft} \]

Now, \[ c_0^2 = \frac{k}{m} \]

\[ = \frac{72 \times 32}{24} \]

\[ c_0^2 = 96 \]

\[ \Rightarrow c_0 = 4\sqrt{6} \]

Now, equation of motion,

\[ x(t) = c_1 \cos c_0 t + c_2 \sin c_0 t \]

\[ x(t) = c_1 \cos 4\sqrt{6} t + c_2 \sin 4\sqrt{6} t \]

Applying \( t = 0 \),

\[ x(0) = -\frac{1}{4} = c_1 \cos (0) + c_2 \sin (0) \]

\[ \Rightarrow c_1 = -\frac{1}{4} \]

\[ x'(t) = -4\sqrt{6} c_1 \sin c_0 t + 4\sqrt{6} c_2 \cos 4\sqrt{6} t \]
Now, \( x'(0) = 0 = -\frac{45\pi}{2} c_1 \sin(\theta) + 4.5\pi c_2 \cos(\theta) \)

\[ \Rightarrow [c_2 = 0] \]

Finally, the equation of motion,

\[ x(t) = -\frac{1}{4} \cos 4\sqrt{5}t \] — solution.

**Ex. 9** A mass weighing 8 pounds is attached to a spring. When set in motion, the spring/mass system exhibits simple harmonic motion. Determine the equation of motion if the spring constant is 1 lb/ft and the mass is initially released from a point 6 inches below the equilibrium position with a downward velocity of \( \frac{3}{2} \) ft/sec. Express the equation of motion in alternative form also.

**Solution:**

\[ W = 8 \text{ lb} \]
\[ m = \frac{W}{g} = \frac{8}{32} \text{ slug} \]
\[ k = 1 \text{ lb/ft} \]
\[ x(0) = 6 \text{ in.} = \frac{1}{2} \text{ ft} \]
\[ x'(0) = \frac{3}{2} \text{ ft/sec} \]

Now, \[ \omega^2 = \frac{k}{m} = \frac{1}{8} \times 32 \]

\[ = 4 \]

\[ \Rightarrow [\omega = 2] \]
Now, equation of motion,

\[ x(t) = g \cos \omega t + c_2 \sin \omega t \]
\[ x(t) = c_1 \cos 2t + c_2 \sin 2t \]

Now applying I.c.

\[ x(0) = \frac{1}{2} = c_1 \cos(0) + c_2 \sin(0) \]

\[ \Rightarrow c_1 = \frac{1}{2} \]

which \[ \Rightarrow \frac{1}{2} \cos \frac{\pi}{2} = \frac{1}{2} \cos \frac{\pi}{2} \]

Now, \[ x'(t) = 2 \left( -c_1 \sin 2t + c_2 \cos 2t \right) \]

\[ x'(0) = 2 \left( -\frac{1}{2} \sin(0) + c_2 \cos(0) \right) = \frac{1}{2} \]

\[ \Rightarrow c_2 = \frac{3}{4} \]

So, now, \[ x(t) = \frac{1}{2} \cos 2t + \frac{3}{4} \sin 2t \] is the solution.

Alternate form:

\[ x(t) = A \sin (\omega t + \phi) \]

\[ A = \sqrt{c_1^2 + c_2^2} = \sqrt{\frac{1}{4} + \frac{9}{16}} = \frac{\sqrt{13}}{4} \]

\[ \phi = \tan^{-1} \left( \frac{\frac{3}{2}}{\frac{1}{2}} \right) = \tan^{-1} \left( \frac{3}{2} \right) \]

\[ \phi = 0.5880 \text{ rad.} \]

So, \[ x(t) = \frac{\sqrt{13}}{4} \sin (2t + 0.5880) \]
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Date: 02/10/2008
SI leader: Monal Kumar Patel.

* SPRING/MASS SYSTEMS:

Case (ii) Free Damped Motion:

FBD:

Now, from Newton's Second Law,

\[ mg - k(x + s) - \beta \ddot{x} = m \ddot{x} \]

\[ mg - kx - \beta \dot{x} = m \ddot{x} \]

So, \[ m \frac{d^2x}{dt^2} + \beta \frac{dx}{dt} + kx = 0 \]

\[ \frac{d^2x}{dt^2} + \frac{\beta}{m} \frac{dx}{dt} + \frac{k}{m} x = 0 \]

Take, \[ co^2 = \frac{k}{m} \] & \[ 2\lambda = \frac{\beta}{m} \]

\[ \frac{d^2x}{dt^2} + 2\lambda \frac{dx}{dt} + co^2 x = 0 \]
The solution of this equation results in 3 different cases:

\[ \lambda^{2} - \omega^{2} > 0 \rightarrow 2 \text{ distinct real roots.} \]

\[ x(t) = e^{-\lambda t} \left( c_1 e^{\sqrt{\lambda^2 - \omega^2} t} + c_2 e^{-\sqrt{\lambda^2 - \omega^2} t} \right) \]

Such a system is known as **overdamped** system because its damping coefficient is larger than the spring constant.

\[ \lambda^{2} - \omega^{2} = 0 \rightarrow \text{Repeated roots} \]

\[ x(t) = e^{-\lambda t} (c_1 + c_2 t) \]

Such a system is known as **critically damped** system. The slight decrease in damping force would result in oscillatory motion for such a case.

\[ \lambda^{2} - \omega^{2} < 0 \rightarrow 2 \text{ complex conjugate roots.} \]

\[ x(t) = e^{-\lambda t} \left( c_1 (\omega \cos \sqrt{\omega^2 - \lambda^2} t + c_2 \sin \sqrt{\omega^2 - \lambda^2} t) \right) \]

This system is known as **underdamped** system as the damping coefficient is small compared to spring constant.
A mass weighing 4 pounds is attached to a spring whose constant is 2 lb/ft. The medium offers a damping force that is numerically equal to the instantaneous velocity. The mass is initially released from a point 1 foot above the equilibrium position with a downward velocity of 8 ft/sec. Determine the time at which the mass passes through the equilibrium position. Find the time at which the mass attains its extreme displacement from the equilibrium position. What is the position of the mass at this instant?

Solution:

\[ W = 4 \text{ lb.} \]
\[ k = 2 \text{ lb/ft} \]
\[ x(0) = -1 \text{ ft} \]
\[ v(0) = 8 \text{ ft/sec} \]
\[ \frac{W}{g} = \frac{4}{32} \]

F_damping = \frac{1}{2}\dot{x}^2

\[ \Rightarrow [B = \frac{1}{2}] \]

Now, \[ 2A = \frac{B}{2m} = \frac{1 \times 32}{4} = 8 \]

\[ 2A = 8 \Rightarrow [A = 4] \]

\[ \omega^2 = \frac{k}{2m} = \frac{2}{4} \times 32 = 16 \]

Therefore, \[ \lambda = \omega^2 = 16 - 16 = 0 \]

Since critically damped system.

Solution

\[ x(t) = e^{-\lambda t} (C_1 + C_2t) \]

\[ x(t) = e^{-4t} (C_1 + C_2t) \]

Applying Initial conditions,

\[ x(0) = -1 = e^{-4(0)} (C_1 + C_2(0)) = C_1 \Rightarrow C_1 = -1 \]
\[ x'(t) = c_2 e^{-4t} - 4 e^{-4t} (c_1 + 2t) \]

\[ x'(0) = 8 = c_2 - 4 c_1 (-1 + 4) \]

\[ \begin{align*}
   c_2 &= 4 \\
   x(0) &= e^{-4t} (-1 + 4t) \tag{1}
\end{align*} \]

Now, 
\[ x'(t) = 4 e^{-4t} - 4 e^{-4t} (-1 + 4t) \]

\[ x'(t) = 8 e^{-4t} - 16 t e^{-4t} \tag{2} \]

At equilibrium condition \( x(t) = 0 \).

From (1), 
\[ 0 = e^{-4t} (-1 + 4t) \]

\[ 4t = 1 \]

\[ t = \frac{1}{4} \text{ sec} \]

For the extreme displacement \( x(t) = 0 \)

From (2), 
\[ 0 = 8 e^{-4t} - 16 t e^{-4t} \]

\[ 8 - 16t = 0 \]

\[ 16t = 8 \]

\[ t = \frac{1}{2} \text{ sec} \]

For \( t = \frac{1}{2} \text{ sec} \) the extreme displacement,

\[ x(t) = e^{-4 \left( \frac{1}{2} \right)} \left( -1 + 4 \left( \frac{1}{2} \right) \right) \]

\[ = e^{-2} \text{ feet} \]

\[ x \left( \frac{1}{2} \right) = 0.138 \text{ feet} \]

\[ = \text{extreme displacement} \]
MAE 3360

SI Session:

Date: 03/12/2008

SI Leader: Monakumar Patel

**SPRING/Mass System:**

**Case (iii) Driven System:**

If the external force \( f(t) \) is considered into account then the equation of motion from Newton's second law,

\[
\frac{d^2 x}{dt^2} + \beta \frac{dx}{dt} + \omega^2 x = f(t)
\]

which is a non-homogeneous equation and can be solved as by using undetermined coefficient method or variation of parameter method to solve the equation.

**Example 29**

A mass weighing 16 pounds stretches a spring \( 8/3 \) feet. The mass is initially released from rest from a point \( 2 \) ft below the equilibrium position, and the subsequent motion takes place in a medium that offers a damping force numerically equal to \( \frac{1}{2} \) the instantaneous velocity. Find the equation of motion if the mass is driven by an external force equal to \( f(t) = 10 \cos 3t \).
From given data...

\[ W = 16 \text{ lb} \quad \Rightarrow \quad W = \frac{16}{g} = \frac{16}{32} \text{ slugs} \]

\[ S = \frac{8}{3} \text{ ft} \]

\[ F_{\text{damping}} = \frac{1}{2} x^2 \quad \text{so, } \beta = \frac{1}{2} \]

\[ x(0) = \frac{8}{3} \text{ ft} \]

\[ x'(0) = 0 \]

\[ f(t) = 10 \cos 3t \text{ lb} \]

\[ \text{Now, } \text{using } k \text{ and } g \]

\[ \Rightarrow \quad 16 = k \left( \frac{8}{3} \right) \]

\[ \Rightarrow \quad k = \frac{18}{8} \text{ slugs/ft} \]

\[
\begin{align*}
\omega^2 &= \frac{k}{m} \\
&= \frac{G \times 32}{16} \\
\omega^2 &= \frac{32}{16} \\
2\lambda &= \frac{B}{m} = \frac{10}{16} = \frac{5}{8} \\
\lambda &= \frac{5}{16} \text{ slugs/second} \\
f(t) &= \frac{dt}{2} \\
f(t) &= 20 \cos 3t \\
\end{align*}
\]

\[ \text{Now, rearrange the equation of the system can be written as} ... \]

\[ \frac{d^2 x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x = 20 \cos 3t \]

**Homogeneous case** \( (x_h) \):

\[ \frac{d^2 x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x = 0 \]

\[ \Rightarrow \quad \frac{d^2 x}{dt^2} + \frac{dx}{dt} + 12 x = 0 \]

**Characteristic equation** \( M^2 + M + 12 = 0 \)

\[ M = \frac{-1 \pm \sqrt{1 - 48}}{2} = \frac{-1 \pm \sqrt{-47}}{2} i \]
so, \( x_h(t) = e^{-1/2 t} \left( c_1 \cos \frac{\sqrt{47}}{2} t + c_2 \sin \frac{\sqrt{47}}{2} t \right) \)

Now, **Non-Homogeneous Part** \((x_p(t))\):

Let, \( x_p = A \cos 3t + B \sin 3t \)

\[
x_p' = -3A \sin 3t + 3B \cos 3t
\]

\[
x_p'' = -9A \cos 3t - 9B \sin 3t
\]

Now, \( x_p'' + x_p' + 12x_p = 20 \cos 3t \)

So, \(-9A \cos 3t - 9B \sin 3t + -3A \sin 3t + 3B \cos 3t + 12A \cos 3t + 12B \sin 3t = 20 \cos 3t \)

\[
= (3A + 3B) \cos 3t + (3B - 3A) \sin 3t = 20 \cos 3t
\]

Now composing both sides,

\[
3A + 3B = 20
\]

\[
3B - 3A = 0
\]

So, solving above two equations, we have \( B = \frac{10}{3} \) and \( A = \frac{10}{3} \).

\[
x_p(t) = \frac{10}{3} \left( \cos 3t + \sin 3t \right)
\]

Finally, general solution,

\[
x(t) = x_h(t) + x_p(t)
\]

\[
x(t) = e^{-1/2 t} \left( c_1 \cos \frac{\sqrt{47}}{2} t + c_2 \sin \frac{\sqrt{47}}{2} t + \frac{10}{3} \left( \cos 3t + \sin 3t \right) \right)
\]

\[
+ 10/3 \left( \cos 3t + \sin 3t \right)
\]
\[ x(t) = e^{-\frac{47}{2}t} \left( -\frac{47}{2} c_1 \sin \frac{\sqrt{47}}{2} t + \frac{47}{2} c_2 \cos \frac{\sqrt{47}}{2} t \right) \]

\[ -\frac{1}{2} e^{-\frac{47}{2}t} \left( c_1 \cos \frac{\sqrt{47}}{2} t + c_2 \sin \frac{\sqrt{47}}{2} t \right) \]

\[ + \frac{10}{3} (-3 \sin 3t + 3 \cos 3t) \]

Case, applying I.C.'s:

\[ x(0) = 2 = e^0 \left( c_1 (1) + c_2 (0) \right) + \frac{10}{3} \left( 1 + 0 \right) \]

\[ \Rightarrow c_1 = 2 - \frac{10}{3} = -\frac{4}{3} \]

\[ c_1 = -\frac{4}{3} \]

\[ \therefore \]

\[ x'(0) = 0 = e^0 \left( -\frac{\sqrt{47}}{2} c_1 (0) + \frac{\sqrt{47}}{2} c_2 (1) \right) \]

\[ -\frac{1}{2} e^0 \left( -\frac{4}{3} (1) + c_2 (0) \right) + \frac{10}{2} (-3 (0) + 3 (1)) \]

\[ 0 = \frac{\sqrt{47}}{2} c_2 + \frac{2}{3} + 10 \]

\[ \therefore \frac{\sqrt{47}}{2} c_2 = -\frac{32}{3} \]

\[ c_2 = -\frac{64}{3 \sqrt{47}} \]

\[ \therefore \]

\[ x(t) = e^{-\frac{47}{2}t} \left( -\frac{4}{3} \cos \frac{\sqrt{47}}{2} t - \frac{64}{3 \sqrt{47}} \sin \frac{\sqrt{47}}{2} t \right) \]

\[ + \frac{10}{3} (\cos 3t + \sin 3t) \]

\[ \therefore \text{Solution.} \]
Ex. 31 A mass of 1 slug, when attached to a spring, stretches it 2 feet and then comes to rest in the equilibrium position. Starting at \( t = 0 \), an external force equal to \( f(t) = 8 \sin 4t \) is applied to the system. Find the equation of motion if the surrounding medium offers a damping force numerically equal to 8 times the instantaneous velocity.

Solution: Given Data:

\[
\begin{align*}
m &= 1 \text{ slug} \\
L &= 2 \text{ ft} \\
f(t) &= 8 \sin 4t \\
F_{\text{damping}} &= 8 x' = \beta = 8
\end{align*}
\]

Now, we know that \( mg = KS \)

\[
\begin{align*}
K &= \frac{mg}{S} = \frac{1 \cdot 32}{2} = 16 \text{ lb/ft} \\
\omega^2 &= \frac{K}{m} = \frac{16}{1} = 16 \\
2\lambda &= \frac{\beta}{m} = \frac{8}{1} = 8
\end{align*}
\]

Now the system is defined by the equation:

\[
\begin{align*}
\frac{d^2x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x &= f(t) \\
\Rightarrow \frac{d^2x}{dt^2} + 8 \frac{dx}{dt} + 16x &= 8 \sin 4t
\end{align*}
\]
Now, solution of eqn (5) is as follows....

Homogeneous soln: \( X_h \)

\[
\frac{d^2x}{dt^2} + 8 \frac{dx}{dt} + 16x = 0
\]

\( \text{characteristic equation,} \)

\( M^2 + 8M + 16 = 0 \)

\( \Rightarrow (M + 4)^2 = 0 \)

\( \Rightarrow M = -4 \) a double root

\( \therefore X_h(t) = c_1 e^{-4t} + c_2 te^{-4t} \)

Particular solution \( X_p \):

Let \( X_p(t) = A \sin 4t + B \cos 4t \)

\( \Rightarrow X_p'(t) = 4A \cos 4t - 4B \sin 4t \)

\( \Rightarrow X_p''(t) = -16A \sin 4t - 16B \cos 4t \)

So, now \( X_p'' + 8X_p' + 16X_p = 8 \sin 4t \)

\( \Rightarrow -16A \sin 4t - 16B \cos 4t + 32A \cos 4t - 32B \sin 4t \)

\( + 16A \sin 4t + 16B \cos 4t = 8 \sin 4t \)

Comparing both sides,

\( a) \quad -32B = 8 \quad \Rightarrow B = -\frac{1}{4} \)

\( \quad 82A = 0 \quad \Rightarrow A = 0 \)

\( \therefore X_p(t) = -\frac{1}{4} \cos 4t \)

Now general solution,

\( X(t) = X_h(t) + X_p(t) \)

\( \Rightarrow X(t) = c_1 e^{-4t} + c_2 te^{-4t} \quad -\frac{1}{4} \cos 4t \)
Now, applying initial conditions...

\[ x(0) = 0 = c_1 + \frac{1}{4} \Rightarrow c_1 = \frac{1}{4} \]

\[ x'(t) = -4c_1 e^{-4t} - 4c_2 e^{-4t} + c_2 e^{-4t} + \sin 4t \]

\[ x'(0) = 0 = -4c_1 + c_2 = -1 + c_2 \]

\[ \Rightarrow c_2 = 1 \]

so, \[ x(t) = \frac{1}{4} e^{-4t} + te^{-4t} - \frac{1}{4} \cos 4t \text{ -- solution.} \]

\[ \star \text{ UNDAMPED FORCED MOTION } \]

In the undamped forced motion system there will be no damping force. So, the equation of motion becomes like:

\[ \frac{d^2 x}{dt^2} + \omega^2 x = f(t) \]

\[ (a) \text{ Show that the solution of the initial-value problem } \]

\[ \frac{d^2 x}{dt^2} + \omega^2 x = F_0 \cos \omega t, \quad x(0) = 0, \quad x'(0) = 0 \]

\[ \Rightarrow x(t) = \frac{F_0}{\omega^2 - \gamma^2} (\cos \omega t - \cos \omega t) \]

\[ (b) \text{ Evaluate } \lim_{\gamma \to 0} \frac{F_0}{\omega^2 - \gamma^2} (\cos \omega t - \cos \omega t) \]
Solution (a)

\[
\frac{d^2x}{dt^2} + \omega^2 x = F_0 \cos \gamma t
\]

It is the 2nd order ODE.

Homogeneous solution: \( \chi_h \)

\[
\frac{d^2x}{dt^2} + \omega^2 x = 0
\]

Characteristic equation,

\[
M^2 + \omega^2 = 0
\]

\[\Rightarrow M = \pm i \omega \]

\[
\chi_h(t) = C_1 \cos \omega t + C_2 \sin \omega t
\]

Particular solution \( \chi_p \)

Let, \( \chi_p(t) = A \cos \gamma t + B \sin \gamma t \)

\[
\chi_p' = -\gamma A \sin \gamma t + \gamma B \cos \gamma t
\]

\[
\chi_p'' = -\gamma^2 A \cos \gamma t - \gamma^2 B \sin \gamma t
\]

Now, \( \chi_p'' + \omega^2 \chi_p = F_0 \cos \gamma t \)

\[
-\gamma^2 A \cos \gamma t - \gamma^2 B \sin \gamma t + \omega^2 A \cos \gamma t + \omega^2 B \sin \gamma t = F_0 \cos \gamma t
\]

\[\Rightarrow A (\omega^2 - \gamma^2) \cos \gamma t + B (\omega^2 - \gamma^2) \sin \gamma t = F_0 \cos \gamma t \]

both side comparing \( \omega, \gamma \) eff.,

\[
A (\omega^2 - \gamma^2) = F_0 \quad \Rightarrow \quad A = \frac{F_0}{\omega^2 - \gamma^2}
\]

\[
B (\omega^2 - \gamma^2) = 0 \quad \Rightarrow \quad B = 0
\]
\[ x_p(t) = \frac{F_0}{\omega^2 - \gamma^2} \cos \omega t \]

\[ x(t) = x_h(t) + x_p(t) \]

\[ x(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{F_0}{\omega^2 - \gamma^2} \cos \omega t \]

Now applying initial conditions,

\[ x(0) = 0 \Rightarrow c_1 + \frac{F_0}{\omega^2 - \gamma^2} = 0 \]

\[ c_1 = \frac{F_0}{\gamma^2 - \omega^2} \]

\[ x'(t) = -\omega c_1 \sin \omega t + \omega c_2 \cos \omega t = \frac{F_0 \gamma}{\omega^2 - \gamma^2} \sin \omega t \]

\[ x'(0) = 0 = \omega c_2 - 0 \Rightarrow c_2 = 0 \]

\[ x(t) = \frac{F_0}{\gamma^2 - \omega^2} \left( \cos \omega t - \cos \omega t \right) \]

\[ (b) \lim_{\gamma \to \omega} \frac{F_0}{\omega^2 - \gamma^2} \left( \cos \omega t - \cos \omega t \right) \]

Applying L' Hospital's rule,

\[ \lim_{\gamma \to \omega} \frac{F_0 \rightleftarrows \sin \omega t}{-2\gamma} \]

\[ = \frac{F_0 \left( \sin \omega t \right)}{2\omega} \]