MAE 3360
SI Session

26/03/26/2008.
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* Numerical Methods to solve ODE:

* Modified Euler's Method:

For a given 1st order ODE...

\[ y' = f(x, y) \quad y(x_0) = y_0 \]

For a given step size 'h': The modified Euler's method to find solution is...

\[ x_1 = x_0 + h \]
\[ x_2 = x_1 + h = x_0 + 2h \]
\[ \vdots \]
\[ x_n = x_0 + nh = x_{n-1} + h \]

The 'y' is obtained as follows...

\[ y_{n+1} = y_n + h \cdot f(x_n, y_n) \]
\[ y_{n+1} = y_n + \frac{h}{2} \left[ f(x_n, y_n) + f(x_{n+1}, y_{n+1}^*) \right] \]

Ex 19.1 Use the modified Euler's method to solve

\[ y' = y - x; \quad y(0) = 2 \] on the interval [0, 1] with \( h = 0.1 \)

Solution: Here, \( f(x, y) = y - x \quad f(x_0, y_0) = 2 \)

\[ x_0 = 0; \quad y_0 = 2 \]
\( n = 1: \)
\[
x_1 = x_0 + h = 0.1
\]
\[
y_1^* = y_0 + h \cdot f(x_0, y_0) = 2 + 0.1(2) = 2.2
\]
\[
y_1 = y_0 + \frac{h}{2} \left[ f(x_0, y_0) + f(x_1, y_1^*) \right]
\]
\[
= 2 + \frac{0.1}{2} \left[ 2 + (2.2 - 0.1) \right]
\]
\[
= 2 + 0.1(2 + 2.1)
\]
\[
y_1 = 2.205
\]
\[
f(x_1, y_1) = y_1 - x_1 = 2.105
\]

\( n = 2: \)
\[
x_2 = x_1 + h = 0.2
\]
\[
y_2^* = y_1 + h \cdot f(x_1, y_1) = 2.205 + 0.1(2.105) = 2.4155
\]
\[
f(x_0, y_2^*) = y_2^* - x_2 = 2.3155
\]
\[
y_2 = y_1 + \frac{h}{2} \left[ f(x_1, y_1) + f(x_2, y_2^*) \right]
\]
\[
= 2.205 + \frac{0.1}{2} \left[ 2.105 + 2.3155 \right] = 2.421025
\]
\[
f(x_2, y_2) = 2.221025
\]

\( n = 3: \)
\[
x_3 = x_2 + h = 0.3
\]
\[
y_3^* = y_2 + h \cdot f(x_2, y_2) = 2.421025 + 0.1(2.221025) = 2.643123
\]
\[
y_3^* = 2.643123
\]
\[
f(x_3, y_3^*) = y_3^* - x_3 = 0.843123
\]
\[
y_3 = y_2 + \frac{h}{2} \left[ f(x_2, y_2) + f(x_3, y_3^*) \right]
\]
\[
= 2.421025 + \frac{0.1}{2} \left[ 2.221025 + 2.843123 \right]
\]
\[
y_3 = 2.649202
\]
Now, \( f(x_3, y_3) = y_3 - x_3 = 2.649232 - 0.3 = 2.349232 \). 

\[ n = 4: \quad x_4 = x_3 + h = 0.4 \]
\[ y_4^* = y_3 + h \cdot f(x_3, y_3) = 2.884155 \]
\[ f(x_4, y_4^*) = y_4^* - x_4 = 2.484155 \]
\[ y_4 = y_3 + \frac{h}{2} \left[ f(x_3, y_3) + f(x_4, y_4^*) \right] 
  = 2.649232 + 0.05 \left[ 2.349232 + 2.484155 \right] \]
\[ y_4 = 2.890901 \]
\[ f(x_4, y_4) = y_4 - x_4 = 2.490901 \]

\[ n = 5: \quad x_5 = x_4 + h = 0.5 \]
\[ y_5^* = y_4 + h \cdot f(x_4, y_4) = 3.189991 \]
\[ f(x_5, y_5^*) = y_5^* - x_5 = 2.639991 \]
\[ y_5 = y_4 + \frac{h}{2} \left[ f(x_4, y_4) + f(x_5, y_5^*) \right] 
  = 2.890901 + 0.05 \left[ 2.490901 + 2.639991 \right] \]
\[ y_5 = 3.147446 \]

Now, thus you can proceed up to \( n = 10, (x=1) \) and find out \( y(1) \) =

→ We can form the Table of result as on next page.
<table>
<thead>
<tr>
<th>n</th>
<th>x_n</th>
<th>y_n</th>
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<tr>
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<td>4.714081</td>
</tr>
</tbody>
</table>

*Runge-Kutta Method:*

For given ODE...

\[ y' = f(x, y); \quad y(x_0) = y_0 \quad \text{with} \quad h \text{- stepsize} \]

\[ y_{n+1} = y_n + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}. \]

Where,

- \( k_1 = h \cdot f(x_n, y_n) \)
- \( k_2 = h \cdot f(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}) \)
- \( k_3 = h \cdot f(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}) \)
- \( k_4 = h \cdot f(x_n + h, y_n + k_3) \)
Example: Use the Runge-Kutta method to find $y(0.7)$

for $y' = y^2 + 1; \ y(0.4) = 0.4227930$ with $h = 0.1$

Solution:

$y' = y^2 + 1; \ y(0.4) = 0.4227930$

$f(x, y) = y^2 + 1; \ x_0 = 0.4 \ \ \ \ \ \ y_0 = 0.4227930$

$n = 0:\$

$x_0 = 0.4$

$y_0 = 0.4227930$

$f(x_0, y_0) = y_0^2 + 1 = 1.178754$

$n = 1:\$

$x_1 = x_0 + h = 0.4 + 0.1 = 0.5$

$k_1 = h \cdot f(x_0, y_0) = 0.1 \cdot (1.178754) = 0.1178754$

$k_2 = h \cdot f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = h \cdot f\left(0.4 + 0.05, \ 0.4227930 + \frac{0.1178754}{2}\right)$

$= 0.1 \cdot f(0.45, 0.4817307)$

$k_2 = 0.123207$

$k_3 = h \cdot f(x_0 + \frac{h}{2}, y_0 + k_2)$

$k_3 = 0.123464$

$k_4 = h \cdot f(x_0 + h, y_0 + k_3)$

$k_4 = 0.129840$

Now, $y_1 = y_0 + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$

$= 0.4227930 + \frac{0.1178754 + 2(0.123207 + 0.123464) + 0.129840}{6}$

$= 0.546203$

$f(x_1, y_1) = y_1^2 + 1 = 1.298447$
\[ n = 2: \quad x_2 = x_1 + h = 0.6 \]

\[ k_1 = h \cdot f(x_1, y_1) = 0.1 \cdot (1.298447) = 0.1298447 \]

\[ k_2 = h \cdot f(x_1 + h/2, y_1 + k_1/2) = 0.1 \cdot f(0.55, 0.611225) \\
    = 0.137360 \]

\[ k_3 = h \cdot f(x_1 + h/2, y_1 + k_2/2) = 0.1 \cdot f(0.55, 0.614983) \\
    = 0.137820 \]

\[ k_4 = h \cdot f(x_1 + h, y_1 + k_3) = 0.1 \cdot f(0.6, 0.684123) \\
    = 0.146802 \]

Now, \[ y_2 = y_1 + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} \]

\[ = 0.546306 + 0.1298447 \cdot 1.2 (0.137360 + 0.137820) + 0.146802 \]

\[ y_2 = 0.684140 \]

\[ f(x_2, y_2) = y_2^2 + 1 = 1.468048 \]

\[ n = 3: \quad x_3 = x_2 + h = 0.7 \]

\[ k_1 = h \cdot f(x_2, y_2) = 0.1 \cdot (1.468048) = 0.1468048 \]

\[ k_2 = h \cdot f(x_2 + h/2, y_2 + k_1/2) = 0.1 \cdot f(0.65, 0.7575424) \\
    = 0.157387 \]

\[ k_3 = h \cdot f(x_2 + h/2, y_2 + k_2/2) = 0.1 \cdot f(0.65, 0.76282A) \\
    = 0.158192 \]
\[ k_4 = h \cdot f(x_3 + \frac{h}{2}, y_2 + \frac{k_3}{6}) = 0.1 \cdot f(0.7, 0.842382) = 0.170952 \]

Now, \[ y_3 = y_2 + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} = 0.684140 + \frac{0.1468348 + 4(0.157387 + 0.158192) + 0.170952}{6} \]

\[ y_3 = 0.842293 \]

\[ y(0.7) = 0.842293 \]

*Solution of Second Order ODE*

We can also solve the second order ODE by using numerical methods. For that first we have to obtain first order ODE from given second order ODE. We can use Euler's Method or Runga-Kutta method to solve such problems. The following example shows the explanation of such methods.

Ex: Use Euler's method to solve \[ y'' - 3y' + 2y = 0; \] \( y(0) = -1 \) \& \( y'(0) = 0 \) on the interval \([0, 1]\) with \( h = 0.1 \)

Solution: \[ y'' - 3y' + 2y = 0 \]

Let, \( y' = z = f(x, y, z) \)

So, eqn becomes, \[ z' - 3z + 2y = 0. \]

\[ z' = 3z - 2y; \quad y(0) = -1 \]

\[ z(0) = 0 \]
From given initial conditions,

\[ x_0 = 0, \quad y_0 = -1 \quad \text{and} \quad z_0 = 0. \]

Now, \( n = 0 \):

\[ y_0' = f(x_0, y_0, z_0) = z_0 = 0. \]

\[ z_0' = g(x_0, y_0, z_0) = 3z_0 - 2y_0 = 3(0) - 2(-1) = 2. \]

\[ y_1 = y_0 + h y_0' = -1 + 0.1(0) = -1. \]

\[ z_1 = z_0 + h z_0' = 0 + 0.1(2) = 0.2. \]

\( n = 1 \):

\[ y_1' = f(x_1, y_1, z_1) = z_1 = 0.2. \]

\[ z_1' = g(x_1, y_1, z_1) = 3z_1 - 2y_1 = 3(0.2) - 2(-1) = 2.6. \]

\[ y_2 = y_1 + h y_1' = -1 + 0.1(0.2) = -0.78. \]

\[ z_2 = z_1 + h z_1' = 0.2 + 0.1(2.6) = 0.46. \]

\( n = 2 \):

\[ y_2' = f(x_2, y_2, z_2) = z_2 = 0.46. \]

\[ z_2' = g(x_2, y_2, z_2) = 3z_2 - 2y_2 = 3(0.46) - 2(-0.78) = 3.34. \]

\[ y_3 = y_2 + h y_2' = -0.78 + 0.1(0.46) = -0.934. \]

\[ z_3 = z_2 + h z_2' = 0.46 + 0.1(3.34) = 0.794. \]

\( n = 3 \):

\[ y_3' = f(x_3, y_3, z_3) = z_3 = 0.794. \]

\[ z_3' = g(x_3, y_3, z_3) = 3z_3 - 2y_3 = 3(0.794) - 2(-0.934) = 4.85. \]

\[ y_4 = y_3 + h y_3' = -0.934 + 0.1(0.794) = -0.8546. \]

\[ z_4 = z_3 + h z_3' = 0.794 + 0.1(4.85) = 1.219. \]

\( n = 4 \):

\[ y_4' = f(x_4, y_4, z_4) = z_4 = 1.219. \]

\[ z_4' = g(x_4, y_4, z_4) = 3z_4 - 2y_4 = 3(1.219) - 2(-0.8546) = 5.3662. \]
\[ y_5 = y_4 + h y'_4 = -0.8546 + 0.1(1.219) = -0.7327 \]
\[ z_5 = z_4 + h z'_4 = 1.219 + 0.1(5.3662) = 1.75562 \]

\[ y'_5 = f(x_5, y_5, z_5) = z_5 = 1.75562 \]

\[ z'_5 = g(x_5, y_5, z_5) = 3z_5 - y'_5 = 6.73226 \]
\[ y_6 = y_5 + h y'_5 = -0.7327 + (0.1)(1.75562) \]
\[ y_6 = -0.557188 \]

\[ z_6 = z_5 + h z'_5 = 1.75562 + 0.1(6.73226) = 2.428846 \]

Thus, you can find the remaining terms up to \( i = 10 \) and we can obtain the value in \([0,1]\). Which forms the table as follows.

<table>
<thead>
<tr>
<th>( X_n )</th>
<th>( Y_n )</th>
<th>( Z_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>-1.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.1</td>
<td>-1.0000</td>
<td>0.2000</td>
</tr>
<tr>
<td>0.2</td>
<td>-0.9800</td>
<td>0.4600</td>
</tr>
<tr>
<td>0.3</td>
<td>-0.9340</td>
<td>0.7440</td>
</tr>
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<td>0.7</td>
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<td>7.1960</td>
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</table>
* Runge-Kutta method.

For a second order ODE...

\[ y' = f(x, y, z) \]

\[ z' = g(x, y, z) \]

The Runge-Kutta method is used as follows...

\[ y_{n+1} = y_n + \frac{1}{6} \left( k_1 + 2k_2 + 2k_3 + k_4 \right) \]

\[ z_{n+1} = z_n + \frac{1}{6} \left( l_1 + 2l_2 + 2l_3 + l_4 \right) \]

where,

\[ k_1 = h \cdot f(x_n, y_n, z_n) \]

\[ l_1 = g \cdot g(x_n, y_n, z_n) \]

\[ k_2 = h \cdot f(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}, z_n + \frac{l_1}{2}) \]

\[ l_2 = h \cdot g(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}, z_n + \frac{l_1}{2}) \]

\[ k_3 = h \cdot f(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}, z_n + \frac{l_2}{2}) \]

\[ l_3 = h \cdot g(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}, z_n + \frac{l_2}{2}) \]

\[ k_4 = h \cdot f(x_n + h, y_n + k_3, z_n + l_3) \]

\[ l_4 = h \cdot g(x_n + h, y_n + k_3, z_n + l_3) \]

Note: You can try Runge-Kutta method to solve the given example with using above formulae.