SI Session
MAE 3360

DT: 02/10/2008
SI Leaders: Monalkumar Patel.

* SPRING / MASS SYSTEMS *

Case (II) Free Damped Motion:

FBD:

\[ F_{\text{damping}} = \beta x \]

\[ \beta = \text{damping coefficient} \]

Now, from Newton's Second Law,

\[ mg - k(x + x) - \beta x = m \ddot{x} \]

\[ mg - kx - ks - \beta x = m \ddot{x} \]

So,

\[ m \dddot{x} + \beta \ddot{x} + kx = 0 \]

\[ \frac{d^2x}{dt^2} + \frac{\beta}{m} \frac{dx}{dt} + \frac{k}{m} x = 0 \]

Take, \( \omega_0 = \sqrt{\frac{k}{m}} \) & \( 2\lambda = \frac{\beta}{m} \)

\[ \frac{d^2x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega_0^2 x = 0. \]
Solution of this equation results in 3 different cases:

(i) \( \lambda^2 - \omega^2 > 0 \) \( \Rightarrow \) 2 distinct real roots.
\[
x(t) = e^{-\lambda t} \left( c_1 e^{\sqrt{\lambda^2 - \omega^2} t} + c_2 e^{-\sqrt{\lambda^2 - \omega^2} t} \right)
\]

Such a system is known as \underline{overdamped} system because its damping coefficient is larger than the spring constant.

(ii) \( \lambda^2 - \omega^2 = 0 \) \( \Rightarrow \) Repeated roots.
\[
x(t) = e^{-\lambda t} \left( c_1 + c_2 t \right)
\]

Such a system is known as \underline{critically damped} system. The slight decrease in damping force would result in oscillatory motion for such a case.

(iii) \( \lambda^2 - \omega^2 < 0 \) \( \Rightarrow \) 2 complex conjugate roots.
\[
x(t) = e^{-\lambda t} \left( c_1 \cos \sqrt{\omega^2 - \lambda^2} t + c_2 \sin \sqrt{\omega^2 - \lambda^2} t \right)
\]

This system is known as \underline{underdamped} system as the damping coefficient is small compared to spring constant.
A mass weighing 4 pounds is attached to a spring whose constant is 2 lb/ft. The medium offers a damping force that is numerically equal to the instantaneous velocity. The mass is initially released from a point 1 foot above the equilibrium position with an downward velocity of 8 ft/s. Determine the time at which the mass passes through the equilibrium position. Find the time at which the mass attains its extreme displacement from the equilibrium position. What is the position of the mass at this instant?

Solution:

\[ W = 4 \text{ lb} \]
\[ x(0) = -1 \text{ ft} \]
\[ x'(0) = 8 \text{ ft/s} \]
\[ m = \frac{W}{g} = \frac{4}{32} \]

\[ F \text{ damping} = x' \]

\[ \Rightarrow \beta = 1 \]

Now, \( 2\lambda = \frac{\beta}{2m} = \frac{1 \times 32}{4} = 8 \)

\[ 2\lambda = 8 \Rightarrow \lambda = 4 \]

\[ \omega^2 = \frac{k}{m} = \frac{2 \times 32}{4} = 16 \]

\[ \text{Now, } x^2 - \omega^2 = 16 - 16 = 0 \]

So it's critically damped system.

Solution

\[ x(t) = e^{-\lambda t} (C_1 + C_2 t) \]

\[ x(t) = e^{-4t} (C_1 + C_2 t) \]

Applying Initial conditions,

\[ x(0) = -1 = e^{-4(0)} (C_1 + C_2(0)) = C_1 \Rightarrow C_1 = -1 \]
\[ x'(t) = c_2 e^{-4t} - 4 e^{-4t} (c_1 + 2t) \]

\[ \Rightarrow x'(0) = 8 = c_2 - 4c_1 (-1 + 0) \]

\[ \Rightarrow c_2 = 4 \]

\[ x(t) = e^{-4t} (-1 + 4t) \]

(1)

Now, \[ x'(t) = 4e^{-4t} - 4e^{-4t} (-1 + 4t) \]

\[ x'(t) = 8e^{-4t} - 16te^{-4t} \]

(2)

At equilibrium condition \( x(t) = 0 \).

From (1), \[ 0 = e^{-4t} (-1 + 4t) \]

\[ \Rightarrow 4t = 1 \]

\[ \Rightarrow t = \frac{1}{4} \text{ sec} \]

For the extreme displacement \( x'(t) = 0 \).

From (2), \[ 0 = 8e^{-4t} - 16te^{-4t} \]

\[ \Rightarrow 8 - 16t = 0 \]

\[ \Rightarrow 16t = 8 \]

\[ \Rightarrow t = \frac{1}{2} \text{ sec} \]

For \( t = \frac{1}{2} \text{ sec} \) the extreme displacement,

\[ x \left( \frac{1}{2} \right) = e^{-4 \left( \frac{1}{2} \right)} \left[ -1 + 4 \left( \frac{1}{2} \right) \right] \]

\[ = e^{-2} \text{ feet} \]

\[ x \left( \frac{1}{2} \right) = 0.135 \text{ feet} \]

\[ \Rightarrow \text{ extreme displacement.} \]