SI Session
MAE 3360

02/20/2008

SI leaders: Monalkumar Patel

A Method of Undetermined Coefficients to solve Non-Homogeneous ODE.

Determine the particular solution to \( L[y] = f(x) \)
for \( f(x) \) as given if the solution to the associated homogeneous equation \( L[y] = 0 \) is \( y_h = e_1 e^{5x} \cos 3x + c_2 e^{5x} \sin 3x \)

1. \( f(x) = x e^{3x} \)
2. \( f(x) = -\cos 3x \)
3. \( f(x) = 5 \cos 5x \)

Find out the general solution for the given differential equations.

11.4b \( y'' - 2y' + y = 4 \cos x \)
17.51 \( y' - y = \sin x + \cos 2x \)
11.52 \( y^{(3)} - 3y'' + 8y' - y = e^x + 1 \)
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Solutions

1. Method of Undetermined Co-efficients:

   Determine the particular solution to $L(y) = \phi(x)$

   $y_p = c_1 e^{5x} \cos 3x + c_2 e^{5x} \sin 3x$

   $\Rightarrow \phi(x) = xe^{3x}$

   \rightarrow Now, family for the given Non-Homogeneous part,

   $xe^{3x} \rightarrow 3xe^{3x} + e^{3x}$

   \therefore $y_p = Ax e^{3x} + Be^{3x}$

   \boxed{$y_p = (Ax + B)e^{3x}$}

2. $\phi(x) = -\cos x$

   \rightarrow The family of the given function is,

   $\cos x \rightarrow (\cos x, \sin x)$

   \boxed{$y_p = A \cos x + B \sin x$}

3. $\phi(x) = 5 \cos \frac{1}{2}x$

   \rightarrow The family for the given function is,

   $\cos \frac{1}{2}x \rightarrow (\cos \frac{1}{2}x, \sin \frac{1}{2}x)$

   \boxed{$y_p = A \cos \frac{1}{2}x + B \sin \frac{1}{2}x$}
Find the general solution of the following equations.

\[ y'' - 2y' + y = 4 \cos x \]

\[ \text{Homogeneous Solution } (y_h): \]

\[ -y'' - 2y' + y = 0 \]

Characteristic Eqn:

\[ m^2 - 2m + 1 = 0 \]

\[ (m-1)^2 = 0 \]

\[ m = 1 \text{ (a double root)} \]

\[ y_h = c_1 e^x + c_2 xe^x \]

\[ \text{Particular Solution } (y_p): \]

\[ y_p = A \cos x + B \sin x \]

\[ y_p' = -A \sin x + B \cos x \]

\[ y_p'' = -A \cos x - B \sin x \]

Now, \[ y_p'' - 2y_p' + y_p = 4 \cos x \]

\[ -A \cos x - B \sin x + 2A \sin x - 2B \cos x + A \cos x + B \sin x = 4 \cos x \]

\[ 2A \sin x - 2B \cos x = 4 \cos x \]

Now comparing both sides,

\[ -2B = 4 \quad \& \quad 2A = 0 \]

\[ B = -2 \quad \& \quad A = 0 \]

\[ y_p = -2 \sin x \]

Now, general solution, \[ y = y_h + y_p \]

\[ y = c_1 e^x + c_2 xe^x - 2 \sin x \]

\[ \text{Solution.} \]
\[ y' - y = \sin x + \cos 2x \]

**Homogeneous Solution**: \((y_h)\):

\[ y' - y = 0 \]

Characteristic equation,

\[ \mu - 1 = 0 \]

:. \[ \mu = 1 \]

:. \[ y_h = c_1 e^x \]

**Particular Solution**: \((y_p)\):

Here, two different families of functions are given. So, we can take the particular solution as the sum of these functions.

\[ y_p = A \sin x + B \cos x \]

\[ y_p = C \sin 2x + D \cos 2x \]

Now, \( y_p = y_{p1} + y_{p2} \)

\[ y_p = A \sin x + B \cos x + C \sin 2x + D \cos 2x \]

So, \( y_p' = A \cos x - B \sin x + 2C \cos 2x - 2D \sin 2x \)

Now, \( y_p' - y_p = \sin x + \cos 2x \)

\[ A \cos x - B \sin x + 2C \cos 2x - 2D \sin 2x - A \sin x - B \cos x \]

\[ - C \sin 2x - D \cos 2x = \sin x + \cos 2x \]

\[ (-A - B) \sin x + (-2D - C) \cos 2x + (A - B) \sin x + (C - D) \cos 2x \]

\[ + (2C - D) \cos 2x = \sin x + \cos 2x \]

Now comparing both sides,

\[ -A - B = 1 \quad (1) \]
\[ -2D - C = 0 \quad (2) \]
\[ A - B = 0 \quad (3) \]
\[ 2C - D = 1 \quad (4) \]
Now, solving (1) and (2), we have,

\[-2A = 1\]

\[\Rightarrow A = -\frac{1}{2} \quad \text{and} \quad B = -\frac{1}{2}\]

Now solving (3) and (4) we have,

\[D = 2C - 1 \quad \Rightarrow -2(2C - 1) - C = 0\]

\[\Rightarrow -4C + 2 - C = 0\]

\[\Rightarrow 5C = 2\]

\[\Rightarrow C = \frac{2}{5}\]

\[\Rightarrow D = -\frac{1}{5}\]

So, now,

\[y_p = -\frac{1}{2} \sin x - \frac{1}{2} \cos x + \frac{2}{5} \sin 2x - \frac{1}{5} \cos 2x\]

So, now general solution

\[y = y_h + y_p\]

\[y = c_1 e^x - \frac{1}{2} \sin x - \frac{1}{2} \cos x + \frac{2}{5} \sin 2x - \frac{1}{5} \cos 2x\]

\[
\underline{(11.52)} \quad y''' - 3y'' + 3y' - y = e^x + 1
\]

\[\Rightarrow \quad \text{Homogeneous Solution:} \quad (y_h)\]

characteristic equation,

\[m^3 - 3m^2 + 3m - 1 = 0\]

\[\Rightarrow (m - 1)^3 = 0\]

\[\Rightarrow m = 1 \quad \text{a triple root}\]

\[y_h = c_1 e^x + c_2 x e^x + c_3 x^2 e^x\]
**Particular Solution** ($Y_p$):

Here, the **Non-Homogeneous function is** $e^x + 1$.

So, we can take, $Y_p = Ae^x \times Y_h = B$.

$Y_p, \quad Y_p = Ae^x + B$.

Now, $Y_p$ contains the terms of homogeneous solution.

$\therefore$ by using modification,

$Y_p, \quad Y_p = Axe^x$

again modify, $Y_p, \quad Y_p = Ax^2 e^x$

again modify, $Y_p, \quad Y_p = Ax^3 e^x$

$\therefore Y_p = Ax^3 e^x + B$.

Now, $Y_p' = 3Ax^2 e^x + Ax^3 e^x$

$Y_p'' = 6Axe^x + 3Ax^2 e^x + 3Ax^2 e^x + Ax^3 e^x$

$Y_p''' = 6Ae^x + 6Axe^x + 6Axe^x + 12Axe^x + 3Axe^x + Ax^3 e^x$

$\therefore Y_p'' = 3Y_p' + 3Y_p^\prime - Y_p = e^x + 1$

$\therefore Ax^3 e^x + 9Ax^2 e^x + 18Axe^x + 6Ae^x$

$- 3Ax^3 e^x - 618Axe^x - 18Axe^x$

$+ 3Ax^2 e^x + 9Ax^2 e^x - Ax^3 e^x - B = e^x + 1$

$\therefore 6A e^x - B = e^x + 1$

Now comparing both sides we have

$6A = 1 \implies A = \frac{1}{6}$

$-B = 1 \implies B = -1$

$\therefore Y_p = \frac{1}{6} x^3 e^x - 1$
Now, general solution of ODE is given as:

\[ y = y_h + y_p \]

\[ \therefore y = c_1 e^x + c_2 xe^x + c_3 x^2 e^x + \frac{1}{6} x^3 e^x - 1 \]

Solution.